PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On the Instability of the Pear-Shaped Figure of Equilibrium of a Rotating Mass of Liquid

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PHILOSOPHICAL TRANSACTIONS.

I. *On the Instability of the Pear-shaped Figure of Equilibrium of a Rotating Mass of Liquid.*

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1. THE main result obtained in this paper is sufficiently indicated by its title. In a recent paper* I showed that the stability of the pear-shaped figure could only be decided on after the figure itself had been calculated as far as terms involving the cube of the parameter e, which is used to measure the deviation of the pear-shaped figure from the critical Jacobian ellipsoid. In the present paper these third-order terms are calculated, and the pear-shaped figure is definitely shown to be unstable.

A statement of the results obtained, and a discussion of their bearing on the wider question of which this problem forms a part, will be found at the end of the paper (§§ 23–27).

2. The discussion has to begin with a determination of the potential of a distorted ellipsoid, carried as far as the third order of the small quantities involved. With a view to shortening very lengthy computations, it is convenient to arrange the algebraic solution in a form somewhat different from that previously given. The solution now given can readily be extended to any order of small quantities, and appears to lead to the most concise series of computations for terms of all degrees above the second.

Potential of a Distorted Ellipsoid carried to the Terms of Third and Higher Orders.

3. As before, the undisturbed ellipsoid (which will ultimately be supposed to be the critical Jacobian) is taken to be the surface $\lambda = 0$ in the family of surfaces

$$f \equiv \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - 1 = 0, \quad \ldots \ldots \quad (1)$$

while the disturbed ellipsoid whose potential we require (which will ultimately be

* "On the Potential of Ellipsoidal Bodies, and the Figures of Equilibrium of Rotating Liquid Masses," 'Phil. Trans.,' A, vol. 215, p. 27.

supposed to be the pear-shaped figure) is taken to be the surface $\lambda = 0$ in the more general family,

$$f+\phi \equiv \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} + \phi - 1 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Here ϕ is a function of x, y, z, and λ, different values for ϕ representing different distortions of the fundamental ellipsoid.

If V_i, V_0 are the internal and external potentials at x, y, z (or their analytical continuations as explained in the previous paper, § 3), we have seen that V_i, V_0 must be of the form

$$V_i = \int_0^\infty \psi(\lambda)(f+\phi)\, d\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$V_0 = \int_{\lambda'}^\infty \psi(\lambda)(f+\phi)\, d\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where

$$\psi(\lambda) = -\frac{\pi\rho abc}{\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}^{1/2}} = -\frac{\pi\rho abc}{\Delta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In these integrations, x, y, z are treated as constants while λ' is the value of λ at the point x, y, z as determined by equation (2). Furthermore, ϕ must not be selected at random; equations (3) and (4) will only give the true potentials if ϕ is chosen so as to satisfy

$$-\left(\frac{\partial f}{\partial \lambda} + \frac{\partial \phi}{\partial \lambda}\right)\int_0^{\lambda'} \psi(\lambda)\, \nabla^2\phi\, d\lambda + \psi(\lambda)\left[4\Sigma\frac{x}{A}\frac{\partial \phi}{\partial x} + 4\frac{\partial \phi}{\partial \lambda} + \Sigma\left(\frac{\partial \phi}{\partial x}\right)^2\right] = 0. \quad . \quad . \quad (6)$$

This equation, as before, is most conveniently solved by a solution

$$\phi = u + fv, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in which u, v must satisfy the equations

$$\int_0^\lambda \psi(\lambda)\, \nabla^2(u+fv)\, d\lambda + 4\psi(\lambda) v = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$4(1+v)\left[\left(\Sigma\frac{x}{A}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial \lambda}\right) + f\left(\Sigma\frac{x}{A}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial \lambda}\right)\right] + \Sigma\left(\frac{\partial u}{\partial x} + f\frac{\partial v}{\partial x}\right)^2 = 0. \quad . \quad . \quad (9)$$

4. We attack the second equation first. As in the previous paper, let us introduce new co-ordinates ξ, η, ζ defined by

$$\xi = \frac{x}{a^2+\lambda} = \frac{x}{A}, \text{ \&c.},$$

and the equation is found to reduce to (*cf.* equation (37) of the previous paper)

$$4(1+v)\left(\frac{\partial u}{\partial \lambda} + f\frac{\partial v}{\partial \lambda}\right) + \Sigma\frac{1}{A^2}\left(\frac{\partial u}{\partial \xi} + f\frac{\partial v}{\partial \xi}\right)^2 = 0.$$

In this equation we may put $f+\phi = 0$, or, since $\phi = u+fv$, we may put

$$f = -\frac{u}{1+v},$$

although of course it would not be legitimate to equate differential coefficients of these equal quantities. The equation reduces to

$$4\frac{\partial}{\partial\lambda}\left(\frac{u}{1+v}\right) + \Sigma\left[\frac{\partial}{\partial\xi}\left(\frac{u}{1+v}\right)\right]^2 = 0, \quad \ldots \ldots \quad (10)$$

and this may be readily solved in powers of the parameter e on assuming a series of the form

$$\frac{u}{1+v} = eg_1 + e^2g_2 + e^3g_3 + e^4g_4 + \ldots \quad \ldots \ldots \quad (11)$$

Equating coefficients of the different powers of e we obtain

$$\frac{\partial g_1}{\partial\lambda} = 0,$$

$$\frac{\partial g_2}{\partial\lambda} = -\tfrac{1}{4}\Sigma\frac{1}{A^2}\left(\frac{\partial g_1}{\partial\xi}\right)^2,$$

$$\frac{\partial g_3}{\partial\lambda} = -\tfrac{1}{4}\Sigma\frac{1}{A^2}\left(2\frac{\partial g_1}{\partial\xi}\frac{\partial g_2}{\partial\xi}\right), \text{ and so on.}$$

To satisfy the first of these equations, g_1 must be a function of ξ, η, ζ only, say P. To satisfy the remaining equations, write

$$\mathbf{A} = \frac{1}{a^2} - \frac{1}{A}, \text{ \&c., so that } \frac{\partial\mathbf{A}}{\partial\lambda} = \frac{1}{A^2}, \text{ \&c.}$$

Then if P_ξ is written for $\partial P/\partial\xi$, the equation for g_2 is

$$\frac{\partial g_2}{\partial\lambda} = -\tfrac{1}{4}\Sigma\frac{\partial\mathbf{A}}{\partial\lambda}P_\xi^2,$$

of which the solution is

$$g_2 = -\tfrac{1}{4}(\mathbf{A}P_\xi^2 + \mathbf{B}P_\eta^2 + \mathbf{C}P_\zeta^2) + Q, \quad \ldots \ldots \quad (12)$$

where Q is a function of ξ, η, ζ only. Proceeding in the same way, we find

$$g_3 = \tfrac{1}{8}(\mathbf{A}^2P_\xi^2P_{\xi\xi} + \ldots + 2\mathbf{BC}P_\eta P_\zeta P_{\eta\zeta} + \ldots) - \tfrac{1}{2}(\mathbf{A}P_\xi Q_\xi + \ldots) + R. \quad . \quad . \quad (13)$$

where R is another function of ξ, η, ζ only, and so on.

5. We now return to equation (8). By a transformation given in the previous paper (p. 37), it is found that this is satisfied by taking

$$v = 0 \quad \text{when} \quad \lambda = 0, \qquad (14)$$

and

$$\nabla^2 u + f\nabla^2 v + 4\left[\Sigma \frac{x}{A}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial \lambda}\right] \equiv \Delta \frac{\partial \sigma}{\partial \lambda}, \qquad (15)$$

where σ may be any function of x, y, z and λ, which vanishes when $\lambda = 0$ and when $\lambda = \lambda'$. For equation (15) we may try provisionally a solution

$$v = w + fw' + f^2 w'' + \ldots + f^n w_n + \ldots \qquad (16)$$

where w, w', w'' ... are quantities satisfying

$$4\left(\Sigma \frac{x}{A}\frac{\partial w}{\partial x} + \frac{\partial w}{\partial \lambda}\right) = -\nabla^2 u - \Delta \frac{\partial \theta}{\partial \lambda}. \qquad (17)$$

$$4\left(\Sigma \frac{x}{A}\frac{\partial w'}{\partial x} + \frac{\partial w'}{\partial \lambda}\right) = -\tfrac{1}{2}\nabla^2 w, \ldots \qquad (18)$$

$$4\left(\Sigma \frac{x}{A}\frac{\partial w_n}{\partial x} + \frac{\partial w_n}{\partial \lambda}\right) = -\frac{1}{n+1}\nabla^2 w_{n-1}, \text{ \&c.} \qquad (19)$$

After a good deal of simplification, the left-hand member of equation (15) is found to reduce to

$$\nabla^2 u + f\nabla^2 v + 4\left(\Sigma \frac{x}{A}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial \lambda}\right) = -\Delta \frac{\partial}{\partial \lambda}\left[\theta + \frac{4}{\Delta}(fw' + 2f^2 w'' + \ldots + nf^n w_n + \ldots)\right]. \qquad (20)$$

The quantity θ is so far undetermined. Let it be given by

$$\theta = \frac{4}{\Delta}\left(\frac{u}{1+v}\right)(w' + 2fw'' + \ldots + nf^{n-1}w_n + \ldots), \qquad (21)$$

so that the quantity in the square bracket on the right of equation (20) becomes

$$\left[\frac{4}{\Delta}\left(f + \frac{u}{1+v}\right)(w' + 2fw'' + \ldots + nf^{n-1}w_n + \ldots)\right]. \qquad (22)$$

When $\lambda = \lambda'$, this vanishes through the factor $f + \dfrac{u}{1+v}$. It will vanish when $\lambda = 0$ through the last factor if we make

$$w' = w'' = w''' = \ldots = 0, \quad \text{when} \quad \lambda = 0. \qquad (23)$$

If this last condition is satisfied, expression (22) satisfies completely the conditions which have to be satisfied by σ in equation (15). Hence the value of v given by equation (16) will be a solution of equation (15). Moreover, if w', w'', w''' ... all vanish when $\lambda = 0$, v will also vanish when $\lambda = 0$, so that equation (14) will also be satisfied. It follows that equations (16) to (19), with (23), contain a complete solution of equation (8).

6. These equations can be solved in powers of the parameter e. Let us assume for u, v expansions in the form

$$u = eu_1+e^2u_2+e^3u_3+e^4u_4+\dots ,$$

$$v = ev_1+e^2v_2+e^3v_3+e^4v_4+\dots ,$$

and for w, w',w'', ... &c., expansions

$$w = ew_1+e^2w_2+e^3w_3+e^4w_4+\dots ,$$

$$w' = ew'_1+e^2w'_2+e^3w'_3+e^4w'_4+\dots ,$$

The coefficients in the expansions of u, v are of course not independent of those in the expansion (11) already assumed for $u/(1+v)$. We find easily enough the relations

$$u_1 = g_1 = \mathrm{P},$$

$$u_2 = g_2+v_1g_1,$$

$$u_3 = g_3+v_1g_2+v_2g_1,$$

$$u_4 = g_4+v_1g_3+v_2g_2+v_3g_1, \text{ \&c.}$$

The value of θ (equation (21)) is found to be

$$\theta = \frac{4}{\Delta}\left[e^2g_1(w'_1+2fw''_1+3f^2w'''_1+\dots) + e^3\{g_2(w'_1+2fw''_1+3f^2w'''_1+\dots)+g_1(w'_2+2fw''_2+\dots)\}\right].$$

On equating coefficients of different powers of e in equations (17)–(19), we obtain

$$\left\{\begin{array}{ll} 4\left(\Sigma\frac{x}{\mathrm{A}}\frac{\partial w_1}{\partial x}+\frac{\partial w_1}{\partial\lambda}\right) = -\nabla^2 u_1, & (24) \\ 4\left(\Sigma\frac{x}{\mathrm{A}}\frac{\partial w'_1}{\partial x}+\frac{\partial w'_1}{\partial\lambda}\right) = -\tfrac{1}{2}\nabla^2 w_1, \text{ \&c.}, & (25) \end{array}\right.$$

$$\left\{\begin{array}{ll} 4\left(\Sigma\frac{x}{\mathrm{A}}\frac{\partial w_2}{\partial x}+\frac{\partial w_2}{\partial\lambda}\right) = -\nabla^2 u_2-4\Delta\frac{\partial}{\partial\lambda}\left[\frac{g_1}{\Delta}(w'_1+2fw''_1+\dots)\right], & (26) \\ 4\left(\Sigma\frac{x}{\mathrm{A}}\frac{\partial w'_2}{\partial x}+\frac{\partial w'_2}{\partial\lambda}\right) = -\tfrac{1}{2}\nabla^2 w_2, \text{ \&c.} & (27) \end{array}\right.$$

There are an infinite number of such sets of equations, of which we shall need only the set for w_3, w'_3, ... in addition to the two sets above. To simplify the equations as much as possible, let us limit ourselves to the type of distortion which leads to the pear-shaped series of figures of equilibrium. For this, as we saw in the previous paper, u_1 is of degree 3 in ξ, η, ζ, so that w_1 must by equation (24) be of degree unity, and from equation (25) w'_1 must vanish. Similarly u_2 is of degree 4, so that w_2 is of degree 2, w'_2 is of degree zero, and $w''_2 = 0$. Again u_3 will be of degree 5, w_3 of degree 3, w'_3 of degree unity, $w''_3 = 0$; and so on.

The set of equations for w_3, w'_3 ... now reduces to

$$\left\{\begin{array}{ll} 4\left(\Sigma \dfrac{x}{A}\dfrac{\partial w_3}{\partial x}+\dfrac{\partial w_3}{\partial \lambda}\right) = -\nabla^2 u_3 - 4\Delta\dfrac{\partial}{\partial\lambda}\left(\dfrac{g_1 w'_2}{\Delta}\right) & (28) \\ 4\left(\Sigma \dfrac{x}{A}\dfrac{\partial w'_3}{\partial x}+\dfrac{\partial w'_3}{\partial \lambda}\right) = -\tfrac{1}{2}\nabla^2 w_3 & (29) \\ w_3'' = w_3''' = \dots = 0. & \end{array}\right.$$

7. Let us now introduce the operator D, already used in the previous paper (§ 14), defined by

$$D = \mathbf{A}\frac{\partial^2}{\partial\xi^2} + \mathbf{B}\frac{\partial^2}{\partial\eta^2} + \mathbf{C}\frac{\partial^2}{\partial\zeta^2} \qquad (30)$$

By differentiation with respect to λ, we have

$$\frac{\partial D}{\partial\lambda} = \frac{1}{A^2}\frac{\partial^2}{\partial\xi^2} + \frac{1}{B^2}\frac{\partial^2}{\partial\eta^2} + \frac{1}{C^2}\frac{\partial^2}{\partial\zeta^2}$$

so that $\partial D/\partial\lambda$ is the same as ∇^2 transformed into ξ, η, ζ co-ordinates. We can now solve equations (24) to (29) at once by transforming into ξ, η, ζ co-ordinates.

Equation (24) becomes in ξ, η, ζ co-ordinates

$$4\frac{\partial w_1}{\partial\lambda} = -\frac{\partial D}{\partial\lambda}u_1 = -\frac{\partial D}{\partial\lambda}P,$$

and since P is a function of ξ, η, ζ only, this has the integral

$$w_1 = -\tfrac{1}{4}DP. \qquad (31)$$

No constant of integration must be added, for D vanishes when $\lambda = 0$ and w_1 must also be made to vanish when $\lambda = 0$ (*cf.* equations (23)). Thus equation (31) gives the true value of w_1 and we have also seen above that $w'_1 = w''_2 = \dots = 0$.

The value of v_1 is accordingly

$$v_1 = w_1 = -\tfrac{1}{4}DP = -\tfrac{1}{4}(\mathbf{A}P_{\xi\xi}+\mathbf{B}P_{\eta\eta}+\mathbf{C}P_{\zeta\zeta}). \qquad (32)$$

From §§ 6 and 4, we now have as the value of u_2,

$$\begin{aligned} u_2 &= g_2 + v_1 g_1 \\ &= -\tfrac{1}{4}(\mathbf{A}P_\xi^2 + \mathbf{B}P_\eta^2 + \mathbf{C}P_\zeta^2) + Q - \tfrac{1}{4}P(\mathbf{A}P_{\xi\xi} + \mathbf{B}P_{\eta\eta} + \mathbf{C}P_{\zeta\zeta}) \\ &= -\tfrac{1}{8}DP^2 + Q. \qquad (33) \end{aligned}$$

The value of w_2 can next be found from equation (26). The right-hand member reduces to $-\nabla^2 u_2$, and the equation, expressed in ξ, η, ζ co-ordinates, becomes

$$\begin{aligned} 4\frac{\partial w_2}{\partial\lambda} &= -\frac{\partial D}{\partial\lambda}u_2 \\ &= \tfrac{1}{8}\frac{\partial D}{\partial\lambda}DP^2 - \frac{\partial D}{\partial\lambda}Q, \end{aligned}$$

of which the solution is

$$w_2 = \tfrac{1}{64}D^2P^2 - \tfrac{1}{4}DQ \qquad (34)$$

and similarly equation (27) leads to

$$w'_2 = -\tfrac{1}{1536}D^3P^2 + \tfrac{1}{64}D^2Q. \qquad (35)$$

These values for w_2 and w'_2 are identical with those obtained in the earlier paper, although obtained by a slightly different method. We now proceed to the third order terms.

On substituting for u_3 its value as obtained in § 6, we find, in place of equation (28),

$$-4\left(\Sigma\frac{x}{A}\frac{\partial w_3}{\partial x} + \frac{\partial w_3}{\partial\lambda}\right) = \nabla^2(g_3 + v_1 g_2) + \nabla^2(v_2 g_1) + 4\Delta\frac{\partial}{\partial\lambda}\left(\frac{g_1 w'_2}{\Delta}\right)$$

which, after a good deal of simplification,

$$\begin{aligned} &= \nabla^2(g_3 + v_1 g_2) + \nabla^2(w_2 P) + f w'_2 \nabla^2 P + 4P\frac{\partial w'_2}{\partial\lambda} \\ &= \nabla^2(g_3 + v_1 g_2) + 2\Sigma\frac{1}{A^2}P_\xi\frac{\partial w_2}{\partial\xi} + (w_2 + f w'_2)\nabla^2 P - 4P\frac{\partial w'_2}{\partial\lambda}. \qquad (36) \end{aligned}$$

Introducing the various values which have been obtained for g_3, g_2, v_1, w_2 and w'_2, we find, after simplification, that

$$\begin{aligned} &\nabla^2(g_3 + v_1 g_2) + 2\Sigma\frac{1}{A^2}P_\xi\frac{\partial w_2}{\partial\xi} + w_2\nabla^2 P \\ &\qquad = p\mathbf{A}^2\frac{\partial\mathbf{A}}{\partial\lambda} + q\mathbf{A}^2\frac{\partial\mathbf{B}}{\partial\lambda} + r\mathbf{AB}\frac{\partial\mathbf{A}}{\partial\lambda} + s\mathbf{AB}\frac{\partial\mathbf{C}}{\partial\lambda} + \ldots \\ &\qquad\quad + \varpi\mathbf{A}\frac{\partial\mathbf{A}}{\partial\lambda} + \rho\mathbf{A}\frac{\partial\mathbf{B}}{\partial\lambda} + \sigma\mathbf{A}\frac{\partial\mathbf{C}}{\partial\lambda} + \ldots + \alpha\frac{\partial\mathbf{A}}{\partial\lambda} + \ldots \qquad (37) \end{aligned}$$

where

$$p = \tfrac{1}{32}\left[\frac{\partial^6}{\partial\xi^6}\left(\tfrac{1}{6}P^3\right) - P\frac{\partial^6}{\partial\xi^6}\left(\tfrac{1}{2}P^2\right)\right],$$

$$q = \tfrac{1}{2}r = \tfrac{1}{32}\left[\frac{\partial^6}{\partial\xi^4\partial\eta^2}\left(\tfrac{1}{6}P^3\right) - P\frac{\partial^2}{\partial\xi^4\partial\eta^2}\left(\tfrac{1}{2}P^2\right)\right],$$

$$s = \tfrac{1}{16}\left[\frac{\partial^6}{\partial\xi^2\partial\eta^2\partial\zeta^2}\left(\tfrac{1}{6}P^3\right) - P\frac{\partial^6}{\partial\xi^2\partial\eta^2\partial\zeta^2}\left(\tfrac{1}{2}P^2\right)\right], \text{ \&c.,}$$

$$\rho = -\tfrac{1}{4}\left[\frac{\partial^4}{\partial\xi^2\partial\eta^2}(PQ) - P\frac{\partial^4}{\partial\xi^2\partial\eta^2}Q\right], \text{ \&c.,}$$

$$\alpha = R_{\xi\xi}, \text{ \&c.}$$

Relation (37) can accordingly be put in the form

$$\nabla^2(g_3+v_1g_2) + 2\Sigma\frac{1}{A^2}P_\xi\frac{\partial w_2}{\partial\xi} + w_2\nabla^2 P = \frac{\partial}{\partial\lambda}\left[\tfrac{1}{96}D^3\left(\tfrac{1}{6}P^3\right) - \tfrac{1}{8}D^2(PQ) + DR + 8Pw'_2\right].$$

Equation (36), after transformation to ξ, η, ζ co-ordinates and integration with respect to λ, gives

$$w_3 = -\tfrac{1}{4}\left[\tfrac{1}{96}D^3\left(\tfrac{1}{6}P^3\right) - \tfrac{1}{8}D^2(PQ) + DR\right] - w'_2 P - \tfrac{1}{4}\int_0^{\lambda'} fw'_2\nabla^2 P\,d\lambda. \quad . \quad . \quad (38)$$

Let us now put

$$\int_0^{\lambda} w'_2\nabla^2 P\,d\lambda = \xi\Lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

so that Λ will be a function of λ only. We then have

$$\int_0^{\lambda'} fw'_2\nabla^2 P\,d\lambda = \int_0^{\lambda'} f\xi\frac{\partial\Lambda}{\partial\lambda}\,d\lambda = f\xi\Lambda - \xi(\xi^2+\eta^2+\zeta^2)\int_0^{\lambda'}\Lambda\,d\lambda, \quad . \quad . \quad . \quad (40)$$

whence, on operating with ∇^2,

$$\nabla^2\int_0^{\lambda'} fw'_2\nabla^2 P\,d\lambda = \xi\frac{\partial}{\partial\lambda}\left\{\left(\frac{6}{A}+\frac{2}{B}+\frac{2}{C}\right)\int_0^{\lambda'}\Lambda\,d\lambda\right\}. \quad . \quad . \quad . \quad . \quad (41)$$

Equation (29), transformed to ξ, η, ζ co-ordinates, now becomes

$$\begin{aligned}\frac{\partial w'_3}{\partial\lambda} &= -\tfrac{1}{8}\nabla^2 w_3 \\ &= \tfrac{1}{32}\left[\tfrac{1}{96}\frac{\partial D}{\partial\lambda}D^3\left(\tfrac{1}{6}P^3\right) - \tfrac{1}{8}\frac{\partial D}{\partial\lambda}D^2(PQ) + \frac{\partial D}{\partial\lambda}DR\right] \\ &\quad + \tfrac{1}{8}\frac{\partial}{\partial\lambda}(\xi\Lambda) + \tfrac{1}{32}\xi\frac{\partial}{\partial\lambda}\left\{\left(\frac{6}{A}+\frac{2}{B}+\frac{2}{C}\right)\int_0^{\lambda'}\Lambda\,d\lambda\right\},\end{aligned}$$

giving on integration,

$$w'_3 = \tfrac{1}{32}\left[\tfrac{1}{384}D^4(\tfrac{1}{6}P^3) - \tfrac{1}{24}D^3(PQ) + \tfrac{1}{2}D^2R\right] + \tfrac{1}{8}\xi\Lambda + \tfrac{1}{32}\xi\left(\frac{6}{A} + \frac{2}{B} + \frac{2}{C}\right)\int_0^{\lambda'} \Lambda\, d\lambda.$$

8. The solution is now complete as far as third order terms, but can be expressed in a more convenient form. We have found for the whole value of v_3,

$$\begin{aligned} v_3 &= w_3 + fw'_3 \\ &= -\tfrac{1}{4}\left[\tfrac{1}{96}D^3(\tfrac{1}{6}P^3) - \tfrac{1}{8}D^2(PQ) + DR\right] - w'_2P \\ &\quad + \tfrac{1}{32}f\left[\tfrac{1}{384}D^4(\tfrac{1}{6}P^3) - \tfrac{1}{24}D^3(PQ) + \tfrac{1}{2}D^2R\right] + Z, \end{aligned} \quad \text{. . . . (42)}$$

where Z is formed of terms involving the function Λ, and has its value given by

$$Z = \tfrac{1}{4}\xi(\xi^2+\eta^2+\zeta^2)\int_0^{\lambda'}\Lambda\, d\lambda - \tfrac{1}{8}f\xi\Lambda + \tfrac{1}{8}f\xi\left(\frac{3}{2A} + \frac{1}{2B} + \frac{1}{2C}\right)\int_0^{\lambda'}\Lambda\, d\lambda.$$

The term Z in v_3 gives rise to a term e^3fZ in ϕ, and this in turn leads to a term

$$\psi(\lambda)\, e^3 fZ = -\frac{\pi\rho abc}{\Delta} e^3 fZ,$$

in the function Φ (see §§ 4, 11 of the previous paper), from which the whole solution is derived. Using the value for Z which has just been obtained, we readily find that, in x, y, z co-ordinates,

$$-\frac{\pi\rho abc}{\Delta} e^3 fZ = -\tfrac{1}{8}\pi\rho abc\, e^3 \frac{\partial}{\partial\lambda'}\left\{\frac{f^2x}{\Delta A}\int_0^{\lambda'}\Lambda\, d\lambda\right\}. \quad \text{. . . . . . (43)}$$

We found, however (see footnote to p. 32 of previous paper), that for a given potential problem, the value of Φ is not unique. If any function Φ gives a solution of the potential problem, then it was found that any other function of the form

$$\Phi + \frac{\partial}{\partial\lambda'}\left\{F\int_0^{\lambda'} u\, d\lambda\right\}, \quad \text{. . . . . (44)}$$

will give a solution of the same problem, provided that F is any function of x, y, z, and λ, which vanishes when $\lambda = \lambda'$ (*i.e.*, when x, y, z, and λ are connected by relation (2)), and u is any function of λ whatever. Consistently with these conditions we may take

$$u = \Lambda, \qquad F = \tfrac{1}{8}\pi\rho abc\, e^3 \frac{f(f+\phi)x}{\Delta A},$$

and the new solution (44) becomes

$$\Phi + \tfrac{1}{8}\pi\rho abc\, e^3 \frac{\partial}{\partial\lambda'}\left\{\frac{f(f+\phi)x}{\Delta A}\int_0^{\lambda'}\Lambda\, d\lambda\right\}. \quad \text{. . . . . . . (45)}$$

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In this solution, Φ already contains a term involving Λ, namely that given by expression (43). Combining this with the remaining term in expression (45), we find that the new solution can be put in the form

$$\Phi_0 + \tfrac{1}{8}\pi\rho abc\, e^3 \frac{\partial}{\partial\lambda'}\left\{\frac{f\phi x}{\Delta A}\int_0^{\lambda'} \Lambda\, d\lambda\right\}, \quad \text{(46)}$$

where Φ_0 is the old solution Φ with the terms in Λ omitted. The last term in expression (45), being proportional to $e^3\phi$, is of the fourth order of small quantities. Thus in a solution as far as e^3 only, this term may be omitted, and $\Phi = \Phi_0$ will be a solution. In other words the term Z may be omitted entirely from equation (42), and the remaining terms will still give an accurate solution for v_3.

Omitting this Z-term, we obtain for the third order terms,

$$\begin{aligned} u_3 + fv_3 &= u_3 + f(w_3 + fw'_3) \\ &= \tfrac{1}{32}D^2(\tfrac{1}{6}P^3) - \tfrac{1}{4}D(PQ) + R \\ &\qquad - \tfrac{1}{4}f\{\tfrac{1}{96}D^3(\tfrac{1}{6}P^3) - \tfrac{1}{8}D^2(PQ) + DR\} \\ &\qquad + \tfrac{1}{32}f^2\{\tfrac{1}{384}D^4(\tfrac{1}{6}P^3) - \tfrac{1}{24}D^3(PQ) + \tfrac{1}{2}D^2R\}. \end{aligned} \quad \text{(47)}$$

This completes the solution of the general potential problem.

Potential of the Pear-shaped Figure.

9. Collecting the results obtained in §§ 3–8, we have found that as far as terms of the third order of small quantities, a value of ϕ which satisfies the necessary differential equation (6) is

$$\begin{aligned} \phi &= e(u_1 + fv_1) + e^2(u_2 + fv_2) + e^3(u_3 + fv_3) \\ &= e[P - \tfrac{1}{4}fDP] \\ &\quad + e^2[Q - \tfrac{1}{8}DP^2 + f\{\tfrac{1}{64}D^2(P^2) - \tfrac{1}{4}DQ\} + f^2\{-\tfrac{1}{1536}D^3P^2 + \tfrac{1}{64}D^2Q\}] \\ &\quad + e^3[R - \tfrac{1}{4}fDPQ + \tfrac{1}{192}D^2P^3 - \tfrac{1}{4}f\{\tfrac{1}{576}D^3P^3 - \tfrac{1}{8}D^2PQ + DR\} \\ &\qquad + \tfrac{1}{32}f^2\{\tfrac{1}{2304}D^4P^3 - \tfrac{1}{24}D^3PQ + \tfrac{1}{2}D^2R\}]. \end{aligned} \quad \text{(48)}$$

At the boundary $\lambda = 0$, this value of ϕ reduces to

$$\phi = eP + e^2Q + e^3R, \quad \text{(49)}$$

and since P, Q, R are entirely at our disposal, this is capable of representing the most general displacement possible, as far as the third order of small quantities. We have, however, to save the printing of additional terms, already assumed that P is of a degree not higher than the third in x, y, z and Q of a degree not higher than the fourth. Subject to these limitations, the potential of the ellipsoid deformed in any way can be obtained by inserting the value (48) for ϕ into equations (3) and (4).

10. We shall now limit ourselves to the distortion which was found, in the earlier paper, to lead to the pear-shaped figure of equilibrium. For this particular distortion we found (*cf.* equations (93) and (94) of the earlier paper),

$$\mathrm{P} = \xi\,(\alpha\xi^2+\beta\eta^2+\gamma\zeta^2+\kappa), \qquad (50)$$

$$\mathrm{Q} = \tfrac{1}{4}\,[\,\mathrm{L}\xi^4+\mathrm{M}\eta^4+\mathrm{N}\zeta^4+2l\eta^2\zeta^2+2m\zeta^2\xi^2+2n\xi^2\eta^2+2\,(p\xi^2+q\eta^2+r\zeta^2)+s]. \qquad (51)$$

We shall now assume for R the value

$$\mathrm{R} = \tfrac{1}{4}\,\xi\,[\mathfrak{L}\xi^4+\mathfrak{M}\eta^4+\mathfrak{N}\zeta^4+2\mathfrak{l}\eta^2\zeta^2+2\mathfrak{m}\zeta^2\xi^2+2\mathfrak{n}\xi^2\eta^2+2\,(\mathfrak{p}\xi^2+\mathfrak{q}\eta^2+\mathfrak{r}\zeta^2)+\mathfrak{s}], \qquad (52)$$

this being found adequate to satisfy all the conditions for a figure of equilibrium of a rotating liquid mass.

11. The values of u_1+fv_1 and of u_2+fv_2 were calculated in the former paper (see equations (121), (122) and (132)). In the same way the value of u_3+fv_3 as given by equation (47) is found to be

$$\begin{aligned}
u_3+fv_3 = \;& \xi^5 \;[15\tfrac{3}{4}\mathbf{A}^2\alpha^3 +2\tfrac{5}{8}\mathbf{AB}\alpha^2\beta +2\tfrac{5}{8}\mathbf{AC}\alpha^2\gamma + \tfrac{3}{8}\mathbf{B}^2\alpha\beta^2+\tfrac{1}{4}\mathbf{BC}\alpha\beta\gamma+ \tfrac{3}{8}\mathbf{C}^2\alpha\gamma^2]\\
&+\xi^3\eta^2\;[13\tfrac{1}{8}\mathbf{A}^2\alpha^2\beta +7\tfrac{1}{2}\mathbf{AB}\alpha^2\beta +2\tfrac{1}{2}\mathbf{AC}\alpha\beta\gamma+1\tfrac{7}{8}\mathbf{B}^2\beta^3 +\tfrac{3}{4}\mathbf{BC}\beta^2\gamma + \tfrac{3}{8}\mathbf{C}^2\beta\gamma^2]\\
&+\xi^3\zeta^2\;[13\tfrac{1}{8}\mathbf{A}^2\alpha^2\gamma +2\tfrac{1}{2}\mathbf{AB}\alpha\beta\gamma+7\tfrac{1}{2}\mathbf{AC}\alpha\gamma^2 + \tfrac{3}{8}\mathbf{B}^2\beta^2\gamma+\tfrac{3}{4}\mathbf{BC}\beta\gamma^2 +1\tfrac{7}{8}\mathbf{C}^2\gamma^3]\\
&+\xi\eta^4\;[\,1\tfrac{7}{8}\mathbf{A}^2\alpha\beta^2 +1\tfrac{7}{8}\mathbf{AB}\beta^3 + \tfrac{3}{8}\mathbf{AC}\beta^2\gamma]\\
&+\xi\zeta^4\;[\,1\tfrac{7}{8}\mathbf{A}^2\alpha\gamma^2 + \tfrac{3}{8}\mathbf{AB}\beta\gamma^2 +1\tfrac{7}{8}\mathbf{AC}\gamma^3]\\
&+\xi\eta^2\zeta^2\;[\,3\tfrac{3}{4}\mathbf{A}^2\alpha\beta\gamma+2\tfrac{1}{4}\mathbf{AB}\beta^2\gamma +2\tfrac{1}{4}\mathbf{AC}\beta\gamma^2]\\
&+\xi^3\kappa\;[13\tfrac{1}{8}\mathbf{A}^2\alpha^2 +2\tfrac{1}{2}\mathbf{AB}\alpha\beta +2\tfrac{1}{2}\mathbf{AC}\alpha\gamma + \tfrac{3}{8}\mathbf{B}^2\beta^2 +\tfrac{1}{4}\mathbf{BC}\beta\gamma + \tfrac{3}{8}\mathbf{C}^2\gamma^2]\\
&+\xi\eta^2\kappa\;[\,3\tfrac{3}{4}\mathbf{A}^2\alpha\beta +2\tfrac{1}{4}\mathbf{AB}\beta^2 + \tfrac{3}{4}\mathbf{AC}\beta\gamma]\\
&+\xi\zeta^2\kappa\;[\,3\tfrac{3}{4}\mathbf{A}^2\alpha\gamma + \tfrac{3}{4}\mathbf{AB}\beta\gamma +2\tfrac{1}{4}\mathbf{AC}\gamma^2]\\
&+\xi\kappa^2\;[\,1\tfrac{7}{8}\mathbf{A}^2\alpha + \tfrac{3}{8}\mathbf{AB}\beta + \tfrac{3}{8}\mathbf{AC}\gamma]\\
&-\xi^5\;[\,2\tfrac{5}{8}\mathbf{A}\mathrm{L}\alpha + \tfrac{1}{8}\mathbf{B}\,(\mathrm{L}\beta+2n\alpha) + \tfrac{1}{8}\mathbf{C}\,(\mathrm{L}\gamma+2m\alpha)]\\
&-\xi^3\eta^2\;[\,1\tfrac{1}{4}\mathbf{A}\,(\mathrm{L}\beta+2n\alpha) + \tfrac{3}{4}\mathbf{B}\,(\mathrm{M}\alpha+2n\beta) + \tfrac{1}{4}\mathbf{C}\,(l\alpha+m\beta+n\gamma)]\\
&-\xi^3\zeta^2\;[\,1\tfrac{1}{4}\mathbf{A}\,(\mathrm{L}\gamma+2m\alpha) + \tfrac{1}{4}\mathbf{B}\,(l\alpha+m\beta+n\gamma)+ \tfrac{3}{4}\mathbf{C}\,(\mathrm{N}\alpha+2n\gamma)]\\
&-\xi\eta^4\;[\,\tfrac{3}{8}\mathbf{A}\,(\mathrm{M}\alpha+2n\beta) +1\tfrac{7}{8}\mathbf{B}\mathrm{M}\beta + \tfrac{1}{8}\mathbf{C}\,(\mathrm{M}\gamma+2l\beta)]\\
&-\xi\zeta^4\;[\,\tfrac{3}{8}\mathbf{A}\,(\mathrm{N}\alpha+2n\gamma) + \tfrac{1}{8}\mathbf{B}\,(\mathrm{N}\beta+2l\gamma) +1\tfrac{7}{8}\mathbf{C}\mathrm{N}\gamma]\\
&-\xi\eta^2\zeta^2\;[\,\tfrac{3}{4}\mathbf{A}\,(l\alpha+m\beta+n\gamma)+ \tfrac{3}{4}\mathbf{B}\,(\mathrm{M}\gamma+2l\beta) + \tfrac{3}{4}\mathbf{C}\,(\mathrm{N}\beta+2l\gamma)]\\
&-\xi^3\;[\,1\tfrac{1}{4}\mathbf{A}\,(\mathrm{L}\kappa+2\alpha p) + \tfrac{1}{4}\mathbf{B}\,(n\kappa+\alpha q+\beta p)+ \tfrac{1}{4}\mathbf{C}\,(m\kappa+\gamma p+\alpha r)]\\
&-\xi\eta^2\;[\,\tfrac{3}{4}\mathbf{A}\,(n\kappa+\alpha q+\beta p) + \tfrac{3}{4}\mathbf{B}\,(\mathrm{M}\kappa+2\beta q) + \tfrac{1}{4}\mathbf{C}\,(l\kappa+\gamma q+\beta r)]\\
&-\xi\zeta^2\;[\,\tfrac{3}{4}\mathbf{A}\,(m\kappa+\gamma p+\alpha r) + \tfrac{1}{4}\mathbf{B}\,(l\kappa+\gamma q+\beta r) + \tfrac{3}{4}\mathbf{C}\,(\mathrm{N}\kappa+2\gamma r)]\\
&-\xi\;[\,\tfrac{3}{8}\mathbf{A}\,(2p\kappa+s\alpha) + \tfrac{1}{8}\mathbf{B}\,(2q\kappa+s\beta) + \tfrac{1}{8}\mathbf{C}\,(2r\kappa+s\gamma)]\\
&+\tfrac{1}{4}\xi\;[\mathfrak{L}\xi^4+\mathfrak{M}\eta^4+\mathfrak{N}\zeta^4+2\mathfrak{l}\eta^2\zeta^2+2\mathfrak{m}\zeta^2\xi^2+2\mathfrak{n}\xi^2\eta^2+2\,(\mathfrak{p}\xi^2+\mathfrak{q}\eta^2+\mathfrak{r}\zeta^2)+\mathfrak{s}]
\end{aligned}$$

$$
\begin{aligned}
&-\xi^3 f\ [26\tfrac{1}{4}\mathbf{A}^3\alpha^3+\tfrac{5}{16}\mathbf{B}^3\beta^3+\tfrac{5}{16}\mathbf{C}^3\gamma^3+6\tfrac{9}{16}\mathbf{A}^2\mathbf{B}\alpha^2\beta+6\tfrac{9}{16}\mathbf{A}^2\mathbf{C}\alpha^2\gamma \\
&\qquad +1\tfrac{7}{8}\mathbf{AB}^2\alpha\beta^2+1\tfrac{7}{8}\mathbf{AC}^2\alpha\gamma^2+1\tfrac{1}{4}\mathbf{ABC}\alpha\beta\gamma+\tfrac{3}{16}\mathbf{B}^2\mathbf{C}\beta^2\gamma+\tfrac{3}{16}\mathbf{BC}^2\beta\gamma^2] \\
&-\xi\eta^2 f[6\tfrac{9}{16}\mathbf{A}^3\alpha^2\beta+5\tfrac{5}{8}\mathbf{A}^2\mathbf{B}\alpha\beta^2+1\tfrac{7}{8}\mathbf{A}^2\mathbf{C}\alpha\beta\gamma+2\tfrac{13}{16}\mathbf{AB}^2\beta^3+\tfrac{9}{16}\mathbf{AC}^2\beta\gamma^2+1\tfrac{1}{8}\mathbf{ABC}\beta^2\gamma] \\
&-\xi\zeta^2 f[6\tfrac{9}{16}\mathbf{A}^3\alpha^2\gamma+1\tfrac{7}{8}\mathbf{A}^2\mathbf{B}\alpha\beta\gamma+5\tfrac{5}{8}\mathbf{A}^2\mathbf{C}\alpha\gamma^2+\tfrac{9}{16}\mathbf{AB}^2\beta^2\gamma+2\tfrac{13}{16}\mathbf{AC}^2\gamma^3+1\tfrac{1}{8}\mathbf{ABC}\beta\gamma^2] \\
&-\xi f\ [6\tfrac{9}{16}\mathbf{A}^3\alpha^2\kappa+1\tfrac{7}{8}\mathbf{A}^2\mathbf{B}\alpha\beta\kappa+1\tfrac{7}{8}\mathbf{A}^2\mathbf{C}\alpha\gamma\kappa+\tfrac{9}{16}\mathbf{AB}^2\beta^2\kappa+\tfrac{9}{16}\mathbf{AC}^2\gamma^2\kappa+\tfrac{3}{8}\mathbf{ABC}\beta\gamma\kappa] \\
&+\xi^3 f\ [6\tfrac{9}{16}\mathbf{A}^2\mathrm{L}\alpha+\tfrac{3}{16}\mathbf{B}^2(\mathrm{M}\alpha+2n\beta)+\tfrac{3}{16}\mathbf{C}^2(\mathrm{N}\alpha+2n\gamma)+\tfrac{5}{8}\mathbf{AB}(\mathrm{L}\beta+2n\alpha) \\
&\qquad +\tfrac{5}{8}\mathbf{AC}(\mathrm{L}\gamma+2m\alpha)+\tfrac{1}{8}\mathbf{BC}(l\alpha+m\beta+n\gamma)] \\
&+\xi\eta^2 f[1\tfrac{5}{16}\mathbf{A}^2(\mathrm{L}\beta+2n\alpha)+2\tfrac{13}{16}\mathbf{B}^2\mathrm{M}\beta+\tfrac{3}{16}\mathbf{C}^2(\mathrm{N}\beta+2l\gamma)+1\tfrac{1}{8}\mathbf{AB}(\mathrm{M}\alpha+2n\beta) \\
&\qquad +\tfrac{3}{8}\mathbf{AC}(l\alpha+m\beta+n\gamma)+\tfrac{3}{8}\mathbf{BC}(\mathrm{M}\gamma+2l\beta)] \\
&+\xi\zeta^2 f[1\tfrac{5}{16}\mathbf{A}^2(\mathrm{L}\gamma+2m\alpha)+\tfrac{3}{16}\mathbf{B}^2(\mathrm{M}\gamma+2l\beta)+2\tfrac{13}{16}\mathbf{C}^2\mathrm{N}\gamma+\tfrac{3}{8}\mathbf{AB}(l\alpha+m\beta+n\gamma) \\
&\qquad +1\tfrac{1}{8}\mathbf{AC}(\mathrm{N}\alpha+2n\gamma)+\tfrac{3}{8}\mathbf{BC}(\mathrm{N}\beta+2l\gamma)] \\
&+\xi f\ [1\tfrac{5}{16}\mathbf{A}^2(\mathrm{L}\kappa+2\alpha p)+\tfrac{3}{16}\mathbf{B}^2(\mathrm{M}\kappa+2\beta q)+\tfrac{3}{16}\mathbf{C}^2(\mathrm{N}\kappa+2\gamma r)+\tfrac{3}{8}\mathbf{AB}(n\kappa+\alpha q+\beta p) \\
&\qquad +\tfrac{3}{8}\mathbf{AC}(m\kappa+\gamma p+\alpha r)+\tfrac{1}{8}\mathbf{BC}(l\kappa+\gamma q+\beta r)] \\
&-\xi^3 f\ [1\tfrac{1}{4}\mathfrak{L}\mathbf{A}+\tfrac{1}{4}\mathfrak{n}\mathbf{B}+\tfrac{1}{4}\mathfrak{m}\mathbf{C}] \\
&-\xi\eta^2 f\ [\tfrac{3}{4}\mathfrak{n}\mathbf{A}+\tfrac{3}{4}\mathfrak{M}\mathbf{B}+\tfrac{1}{4}\mathfrak{l}\mathbf{C}] \\
&-\xi\zeta^2 f\ [\tfrac{3}{4}\mathfrak{m}\mathbf{A}+\tfrac{1}{4}\mathfrak{l}\mathbf{B}+\tfrac{3}{4}\mathfrak{N}\mathbf{C}] \\
&-\xi f\ [\tfrac{3}{4}\mathfrak{p}\mathbf{A}+\tfrac{1}{4}\mathfrak{q}\mathbf{B}+\tfrac{1}{4}\mathfrak{r}\mathbf{C}]+\xi f^2\mathrm{G}, \quad \ldots \ldots \quad (53)
\end{aligned}
$$

where f stands for $\mathrm{A}\xi^2+\mathrm{B}\eta^2+\mathrm{C}\zeta^2-1$, and G is given by

$$
\begin{aligned}
\mathrm{G} = \mathbf{A}\,\{&\tfrac{315}{64}\mathbf{A}^3\alpha^3+\tfrac{105}{64}\mathbf{A}^2\mathbf{B}\alpha^2\beta+\tfrac{105}{64}\mathbf{A}^2\mathbf{C}\alpha^2\gamma+\tfrac{45}{64}\mathbf{AB}^2\alpha\beta^2 \\
&+\tfrac{45}{64}\mathbf{AC}^2\alpha\gamma^2+\tfrac{15}{32}\mathbf{ABC}\alpha\beta\gamma+\tfrac{15}{64}\mathbf{B}^3\beta^3+\tfrac{15}{64}\mathbf{C}^3\gamma^3 \\
&+\tfrac{9}{64}\mathbf{B}^2\mathbf{C}\beta^2\gamma+\tfrac{9}{64}\mathbf{BC}^2\beta\gamma^2\} \\
&-\tfrac{105}{64}\mathbf{A}^3\mathrm{L}\alpha-\tfrac{15}{64}\mathbf{A}^2\mathbf{B}(\mathrm{L}\beta+2n\alpha)-\tfrac{15}{64}\mathbf{A}^2\mathbf{C}(\mathrm{L}\gamma+2m\alpha) \\
&-\tfrac{9}{64}\mathbf{AB}^2(\mathrm{M}\alpha+2n\beta)-\tfrac{9}{64}\mathbf{AC}^2(\mathrm{N}\alpha+2n\gamma)-\tfrac{3}{32}\mathbf{ABC}(l\alpha+m\beta+n\gamma) \\
&-\tfrac{15}{64}\mathbf{B}^3\mathrm{M}\beta-\tfrac{15}{64}\mathbf{C}^3\mathrm{N}\gamma-\tfrac{3}{64}\mathbf{B}^2\mathbf{C}(\mathrm{M}\gamma+2l\beta)-\tfrac{3}{64}\mathbf{BC}^2(\mathrm{N}\beta+2l\gamma) \\
&+\tfrac{15}{32}\mathbf{A}^2\mathfrak{L}+\tfrac{3}{32}\mathbf{B}^2\mathfrak{M}+\tfrac{3}{32}\mathbf{C}^2\mathfrak{N}+\tfrac{1}{16}\mathbf{BC}\mathfrak{l}+\tfrac{3}{16}\mathbf{AC}\mathfrak{m}+\tfrac{3}{16}\mathbf{AB}\mathfrak{n}. \quad \ldots \quad (54)
\end{aligned}
$$

12. The potential of the distorted ellipsoid

$$f+e\mathrm{P}+e^2\mathrm{Q}+e^3\mathrm{R} = 0, \quad \ldots \ldots \quad (55)$$

can now be written down as far as terms in e^3. The terms in e and e^2, which of course involve only P and Q, have been calculated in the previous paper. The terms in e^3 in V_b, the potential at the boundary $\lambda = 0$, are (*cf.* equation (3)),

$$e^3\int_0^\infty \psi(\lambda)(u_3+fv_3)\,d\lambda = -\pi\rho abc\,e^3\int_0^\infty \frac{u_3+fv_3}{\Delta}\,d\lambda, \quad \ldots \ldots \quad (56)$$

in which u_3+fv_3 has the value just given in equation (53).

Let us put

$$\int_0^\infty \frac{u_3+fv_3}{\Delta}\,d\lambda = x\left(\mathfrak{c}_{11}x^4+\mathfrak{c}_{22}y^4+\mathfrak{c}_{33}z^4+\mathfrak{c}_{12}x^2y^2+\mathfrak{c}_{31}z^2x^2+\mathfrak{c}_{23}y^2z^2+\mathfrak{d}_1x^2+\mathfrak{d}_2y^2+\mathfrak{d}_3z^2+\mathfrak{d}_4\right). \quad (57)$$

The values of the various coefficients in this expression are found to be as follows :—

$$\begin{aligned}
\mathfrak{c}_{11} = {} & \int_0^\infty \frac{1}{\Delta \mathrm{A}^5}\Big[15\tfrac{3}{4}\mathbf{A}^2\alpha^3+2\tfrac{5}{8}\mathbf{AB}\alpha^2\beta+2\tfrac{5}{8}\mathbf{AC}\alpha^2\gamma+\tfrac{3}{8}\mathbf{B}^2\alpha\beta^2+\tfrac{1}{4}\mathbf{BC}\alpha\beta\gamma \\
& \qquad +\tfrac{3}{8}\mathbf{C}^2\alpha\gamma^2-2\tfrac{5}{8}\mathbf{A}\mathrm{L}\alpha-\tfrac{1}{8}\mathbf{B}\,(\mathrm{L}\beta+2n\alpha)-\tfrac{1}{8}\mathbf{C}\,(\mathrm{L}\gamma+2m\alpha)+\tfrac{1}{4}\mathfrak{L}\Big]\,d\lambda \\
& -\int_0^\infty \frac{1}{\Delta \mathrm{A}^4}\Big[26\tfrac{1}{4}\mathbf{A}^3\alpha^3+\tfrac{5}{16}\mathbf{B}^3\beta^3+\tfrac{5}{16}\mathbf{C}^3\gamma^3+6\tfrac{9}{16}\mathbf{A}^2\mathbf{B}\alpha^2\beta+6\tfrac{9}{16}\mathbf{A}^2\mathbf{C}\alpha^2\gamma \\
& \qquad +1\tfrac{7}{8}\mathbf{AB}^2\alpha\beta^2+1\tfrac{7}{8}\mathbf{AC}^2\alpha\gamma^2+1\tfrac{1}{4}\mathbf{ABC}\alpha\beta\gamma+\tfrac{3}{16}\mathbf{B}^2\mathbf{C}\beta^2\gamma+\tfrac{3}{16}\mathbf{BC}^2\beta\gamma^2\Big]\,d\lambda \\
& +\int_0^\infty \frac{1}{\Delta \mathrm{A}^4}\Big[6\tfrac{9}{16}\mathbf{A}^2\mathrm{L}\alpha+\tfrac{3}{16}\mathbf{B}^2\,(\mathrm{M}\alpha+2n\beta)+\tfrac{3}{16}\mathbf{C}^2\,(\mathrm{N}\alpha+2n\gamma) \\
& \qquad +\tfrac{5}{8}\mathbf{AB}\,(\mathrm{L}\beta+2n\alpha)+\tfrac{5}{8}\mathbf{AC}\,(\mathrm{L}\gamma+2m\alpha)+\tfrac{1}{8}\mathbf{BC}\,(l\alpha+m\beta+n\gamma)\Big]\,d\lambda \\
& -\int_0^\infty \frac{1}{\Delta \mathrm{A}^4}\Big[1\tfrac{1}{4}\mathbf{A}\mathfrak{L}+\tfrac{1}{4}\mathbf{B}\mathfrak{n}+\tfrac{1}{4}\mathbf{C}\mathfrak{m}\Big]+\int_0^\infty \frac{\mathrm{G}}{\Delta \mathrm{A}^3}\,d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)
\end{aligned}$$

$$\begin{aligned}
\mathfrak{c}_{12} = {} & \int_0^\infty \frac{1}{\Delta \mathrm{A}^3\mathrm{B}^2}\Big[13\tfrac{1}{8}\mathbf{A}^2\alpha^2\beta+7\tfrac{1}{2}\mathbf{AB}\alpha\beta^2+2\tfrac{1}{2}\mathbf{AC}\alpha\beta\gamma+1\tfrac{7}{8}\mathbf{B}^2\beta^3+\tfrac{3}{4}\mathbf{BC}\beta^2\gamma \\
& \qquad +\tfrac{3}{8}\mathbf{C}^2\beta\gamma^2-1\tfrac{1}{4}\mathbf{A}\,(\mathrm{L}\beta+2n\alpha)-\tfrac{3}{4}\mathbf{B}\,(\mathrm{M}\alpha+2n\beta)-\tfrac{1}{4}\mathbf{C}\,(l\alpha+m\beta+n\gamma)+\tfrac{1}{2}\mathfrak{n}\Big]\,d\lambda \\
& -\int_0^\infty \frac{1}{\Delta \mathrm{A}^3\mathrm{B}}\Big[26\tfrac{1}{4}\mathbf{A}^3\alpha^2+\tfrac{5}{16}\mathbf{B}^3\beta^3+\tfrac{5}{16}\mathbf{C}^3\gamma^3+6\tfrac{9}{16}\mathbf{A}^2\mathbf{B}\alpha^2\beta+6\tfrac{9}{16}\mathbf{A}^2\mathbf{C}\alpha^2\gamma \\
& \qquad +1\tfrac{7}{8}\mathbf{AB}^2\alpha\beta^2+1\tfrac{7}{8}\mathbf{AC}^2\alpha\gamma^2+1\tfrac{1}{4}\mathbf{ABC}\alpha\beta\gamma+\tfrac{3}{16}\mathbf{B}^2\mathbf{C}\beta^2\gamma+\tfrac{3}{16}\mathbf{BC}^2\beta\gamma^2\Big]\,d\lambda \\
& -\int_0^\infty \frac{1}{\Delta \mathrm{A}^2\mathrm{B}^2}\Big[6\tfrac{9}{16}\mathbf{A}^3\alpha^2\beta+5\tfrac{5}{8}\mathbf{A}^2\mathbf{B}\alpha\beta^2+1\tfrac{7}{8}\mathbf{A}^2\mathbf{C}\alpha\beta\gamma+2\tfrac{13}{16}\mathbf{AB}^2\beta^3 \\
& \qquad +\tfrac{9}{16}\mathbf{AC}^2\beta\gamma^2+1\tfrac{1}{8}\mathbf{ABC}\beta^2\gamma\Big]\,d\lambda \\
& +\int_0^\infty \frac{1}{\Delta \mathrm{A}^2\mathrm{B}^2}\Big[1\tfrac{5}{16}\mathbf{A}^2\,(\mathrm{L}\beta+2n\alpha)+2\tfrac{13}{16}\mathbf{B}^2\mathrm{M}\beta+\tfrac{3}{16}\mathbf{C}^2\,(\mathrm{N}\beta+2l\gamma) \\
& \qquad +1\tfrac{1}{8}\mathbf{AB}\,(\mathrm{M}\alpha+2n\beta)+\tfrac{3}{8}\mathbf{AC}\,(l\alpha+m\beta+n\gamma)+\tfrac{3}{8}\mathbf{BC}\,(\mathrm{M}\gamma+2l\beta)\Big]\,d\lambda \\
& +\int_0^\infty \frac{1}{\Delta \mathrm{A}^3\mathrm{B}}\Big[6\tfrac{9}{16}\mathbf{A}^2\mathrm{L}\alpha+\tfrac{3}{16}\mathbf{B}^2\,(\mathrm{M}\alpha+2n\beta)+\tfrac{3}{16}\mathbf{C}^2\,(\mathrm{N}\alpha+2n\gamma) \\
& \qquad +\tfrac{5}{8}\mathbf{AB}\,(\mathrm{L}\beta+2n\alpha)+\tfrac{5}{8}\mathbf{AC}\,(\mathrm{L}\gamma+2m\alpha)+\tfrac{1}{8}\mathbf{BC}\,(l\alpha+m\beta+n\gamma)\Big]\,d\lambda \\
& -\int_0^\infty \frac{1}{\Delta \mathrm{A}^2\mathrm{B}^2}\Big[\tfrac{3}{4}\mathbf{A}\mathfrak{n}+\tfrac{3}{4}\mathbf{B}\mathfrak{m}+\tfrac{1}{4}\mathbf{C}\mathfrak{l}\Big]\,d\lambda \\
& -\int_0^\infty \frac{1}{\Delta \mathrm{A}^3\mathrm{B}}\Big[1\tfrac{1}{4}\mathbf{A}\mathfrak{L}+\tfrac{1}{4}\mathbf{B}\mathfrak{n}+\tfrac{1}{4}\mathbf{C}\mathfrak{m}\Big]\,d\lambda+2\int_0^\infty \frac{\mathrm{G}}{\Delta \mathrm{A}^2\mathrm{B}}\,d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)
\end{aligned}$$

The value of $\mathfrak{c}_{13}$ can be written down from symmetry.

$$\mathfrak{c}_{22} = \int_0^\infty \frac{1}{\Delta \mathrm{AB}^4}\Big[1\tfrac{7}{8}\mathbf{A}^2\alpha\beta^2 + 1\tfrac{7}{8}\mathbf{AB}\beta^3 + \tfrac{3}{8}\mathbf{AC}\beta^2\gamma - \tfrac{3}{8}\mathbf{A}\,(\mathrm{M}\alpha + 2n\beta) - 1\tfrac{7}{8}\mathbf{B}\mathrm{M}\beta - \tfrac{1}{8}\mathbf{C}\,(\mathrm{M}\gamma + 2l\beta) + \tfrac{1}{4}\mathfrak{M}\Big]\,d\lambda$$

$$- \int_0^\infty \frac{1}{\Delta \mathrm{AB}^3}\Big[6\tfrac{9}{16}\mathbf{A}^3\alpha^2\beta + 5\tfrac{5}{8}\mathbf{A}^2\mathbf{B}\alpha\beta^2 + 1\tfrac{7}{8}\mathbf{A}^2\mathbf{C}\alpha\beta\gamma + 2\tfrac{13}{16}\mathbf{AB}^2\beta^3 + \tfrac{9}{16}\mathbf{AC}^2\beta\gamma^2 + 1\tfrac{1}{8}\mathbf{ABC}\beta^2\gamma\Big]\,d\lambda$$

$$+ \int_0^\infty \frac{1}{\Delta \mathrm{AB}^3}\Big[\tfrac{15}{16}\mathbf{A}^2\,(\mathrm{L}\beta + 2n\alpha) + 2\tfrac{13}{16}\mathbf{B}^2\mathrm{M}\beta + \tfrac{3}{16}\mathbf{C}^2\,(\mathrm{N}\beta + 2l\gamma) + 1\tfrac{1}{8}\mathbf{AB}\,(\mathrm{M}\alpha + 2n\beta) + \tfrac{3}{8}\mathbf{AC}\,(l\alpha + m\beta + n\gamma) + \tfrac{3}{8}\mathbf{BC}\,(\mathrm{M}\gamma + 2l\beta)\Big]\,d\lambda$$

$$- \int_0^\infty \frac{1}{\Delta \mathrm{AB}^3}\Big[\tfrac{3}{4}\mathbf{A}\mathfrak{n} + \tfrac{3}{4}\mathbf{B}\mathfrak{M} + \tfrac{1}{4}\mathbf{C}\mathfrak{l}\Big] + \int_0^\infty \frac{\mathrm{G}}{\Delta \mathrm{AB}^2}\,d\lambda. \quad \ldots \quad (60)$$

The value of $\mathfrak{c}_{33}$ can be written down from symmetry.

$$\mathfrak{c}_{23} = \int_0^\infty \frac{1}{\Delta \mathrm{AB}^2\mathrm{C}^2}\Big[3\tfrac{3}{4}\mathbf{A}^2\alpha\beta\gamma + 2\tfrac{1}{4}\mathbf{AB}\beta^2\gamma + 2\tfrac{1}{4}\mathbf{AC}\beta\gamma^2 - \tfrac{3}{4}\mathbf{A}\,(l\alpha + m\beta + n\gamma) - \tfrac{3}{4}\mathbf{B}\,(\mathrm{M}\gamma + 2l\beta) - \tfrac{3}{4}\mathbf{C}\,(\mathrm{N}\beta + 2l\gamma) + \tfrac{1}{2}\mathfrak{l}\Big]\,d\lambda$$

$$- \int_0^\infty \frac{1}{\Delta \mathrm{AB}^2\mathrm{C}}\Big[6\tfrac{9}{16}\mathbf{A}^3\alpha^2\beta + 5\tfrac{5}{8}\mathbf{A}^2\mathbf{B}\alpha\beta^2 + 1\tfrac{7}{8}\mathbf{A}^2\mathbf{C}\alpha\beta\gamma + 2\tfrac{13}{16}\mathbf{AB}^2\beta^3 + \tfrac{9}{16}\mathbf{AC}^2\beta\gamma^2 + \tfrac{9}{8}\mathbf{ABC}\beta^2\gamma\Big]\,d\lambda$$

$$- \int_0^\infty \frac{1}{\Delta \mathrm{ABC}^2}\Big[6\tfrac{9}{16}\mathbf{A}^3\alpha^2\gamma + 1\tfrac{7}{8}\mathbf{A}^2\mathbf{B}\alpha\beta\gamma + 5\tfrac{5}{8}\mathbf{A}^2\mathbf{C}\alpha\gamma^2 + \tfrac{9}{16}\mathbf{AB}^2\beta^2\gamma + 2\tfrac{13}{16}\mathbf{AC}^2\gamma^3 + \tfrac{9}{8}\mathbf{ABC}\beta\gamma^2\Big]\,d\lambda$$

$$+ \int_0^\infty \frac{1}{\Delta \mathrm{AB}^2\mathrm{C}}\Big[\tfrac{15}{16}\mathbf{A}^2\,(\mathrm{L}\beta + 2n\alpha) + 2\tfrac{13}{16}\mathbf{B}^2\mathrm{M}\beta + \tfrac{3}{16}\mathbf{C}^2\,(\mathrm{N}\beta + 2l\gamma) + \tfrac{9}{8}\mathbf{AB}\,(\mathrm{M}\alpha + 2n\beta) + \tfrac{3}{8}\mathbf{AC}\,(l\alpha + m\beta + n\gamma) + \tfrac{3}{8}\mathbf{BC}\,(\mathrm{M}\gamma + 2l\beta)\Big]\,d\lambda$$

$$+ \int_0^\infty \frac{1}{\Delta \mathrm{ABC}^2}\Big[\tfrac{15}{16}\mathbf{A}^2\,(\mathrm{L}\gamma + 2m\alpha) + \tfrac{3}{16}\mathbf{B}^2\,(\mathrm{M}\gamma + 2l\beta) + 2\tfrac{13}{16}\mathbf{C}^2\mathrm{N}\gamma + \tfrac{3}{8}\mathbf{AB}\,(l\alpha + m\beta + n\gamma) + \tfrac{9}{8}\mathbf{AC}\,(\mathrm{N}\alpha + 2n\gamma) + \tfrac{3}{8}\mathbf{BC}\,(\mathrm{N}\beta + 2l\gamma)\Big]\,d\lambda$$

$$- \int_0^\infty \frac{1}{\Delta \mathrm{ABC}^2}\Big[\tfrac{3}{4}\mathbf{A}\mathfrak{m} + \tfrac{1}{4}\mathbf{B}\mathfrak{l} + \tfrac{3}{4}\mathbf{C}\mathfrak{N}\Big]\,d\lambda - \int_0^\infty \frac{1}{\Delta \mathrm{AB}^2\mathrm{C}}\Big[\tfrac{3}{4}\mathbf{A}\mathfrak{n} + \tfrac{3}{4}\mathbf{B}\mathfrak{M} + \tfrac{1}{4}\mathbf{C}\mathfrak{l}\Big]\,d\lambda$$

$$+ 2\int_0^\infty \frac{\mathrm{G}}{\Delta \mathrm{ABC}}\,d\lambda. \quad \ldots \quad (61)$$

The values of the four coefficients $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, $\mathfrak{D}_4$ are most conveniently expressed in the form :—

$$\mathfrak{D}_1 = \delta_1 - \int_0^\infty \frac{1}{\Delta A^2}\left(\tfrac{3}{4}\mathfrak{p}\mathbf{A}+\tfrac{1}{4}\mathfrak{q}\mathbf{B}+\tfrac{1}{4}\mathfrak{r}\mathbf{C}\right)d\lambda + \tfrac{1}{2}\mathfrak{p}\int_0^\infty \frac{d\lambda}{\Delta A^3} - 2\int_0^\infty \frac{G}{\Delta A^2}\,d\lambda, \quad . \quad . \quad (62)$$

$$\mathfrak{D}_2 = \delta_2 - \int_0^\infty \frac{1}{\Delta AB}\left(\tfrac{3}{4}\mathfrak{p}\mathbf{A}+\tfrac{1}{4}\mathfrak{q}\mathbf{B}+\tfrac{1}{4}\mathfrak{r}\mathbf{C}\right)d\lambda + \tfrac{1}{2}\mathfrak{q}\int_0^\infty \frac{d\lambda}{\Delta AB^2} - 2\int_0^\infty \frac{G}{\Delta AB}\,d\lambda, \quad . \quad . \quad (63)$$

$$\mathfrak{D}_3 = \delta_3 - \int_0^\infty \frac{1}{\Delta AC}\left(\tfrac{3}{4}\mathfrak{p}\mathbf{A}+\tfrac{1}{4}\mathfrak{q}\mathbf{B}+\tfrac{1}{4}\mathfrak{r}\mathbf{C}\right)d\lambda + \tfrac{1}{2}\mathfrak{r}\int_0^\infty \frac{d\lambda}{\Delta AC^2} - 2\int_0^\infty \frac{G}{\Delta AC}\,d\lambda, \quad . \quad . \quad (64)$$

$$\mathfrak{D}_4 = \delta_4 + \int_0^\infty \frac{1}{\Delta A}\left(\tfrac{3}{4}\mathfrak{p}\mathbf{A}+\tfrac{1}{4}\mathfrak{q}\mathbf{B}+\tfrac{1}{4}\mathfrak{r}\mathbf{C}\right)d\lambda + \tfrac{1}{4}\mathfrak{s}\int_0^\infty \frac{d\lambda}{\Delta A} - \int_0^\infty \frac{G}{\Delta A}\,d\lambda, \quad . \quad . \quad (65)$$

where δ_1, δ_2, δ_3, δ_4 are quantities which do not depend on $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$, being given by the following equations :—

$$\begin{aligned}
\delta_1 = \int_0^\infty \frac{1}{\Delta A^3}\Big[&\kappa\left(13\tfrac{1}{8}\mathbf{A}^2\alpha^2+2\tfrac{1}{2}\mathbf{AB}\alpha\beta+2\tfrac{1}{2}\mathbf{AC}\alpha\gamma+\tfrac{3}{8}\mathbf{B}^2\beta^2+\tfrac{1}{4}\mathbf{BC}\beta\gamma+\tfrac{3}{8}\mathbf{C}^2\gamma^2\right)\\
&-1\tfrac{1}{4}\mathbf{A}\left(L\kappa+2\alpha p\right)-\tfrac{1}{4}\mathbf{B}\left(n\kappa+\alpha q+\beta p\right)-\tfrac{1}{4}\mathbf{C}\left(m\kappa+p\gamma+r\alpha\right)\\
&+26\tfrac{1}{4}\mathbf{A}^3\alpha^3+\tfrac{5}{16}\mathbf{B}^3\beta^3+\tfrac{5}{16}\mathbf{C}^3\gamma^3+6\tfrac{9}{16}\mathbf{A}^2\mathbf{B}\alpha^2\beta+6\tfrac{9}{16}\mathbf{A}^2\mathbf{C}\alpha^2\gamma\\
&+1\tfrac{7}{8}\mathbf{AB}^2\alpha\beta^2+1\tfrac{7}{8}\mathbf{AC}^2\alpha\gamma^2+1\tfrac{1}{4}\mathbf{ABC}\alpha\beta\gamma+\tfrac{3}{16}\mathbf{B}^2\mathbf{C}\beta^2\gamma+\tfrac{3}{16}\mathbf{BC}^2\beta\gamma^2\Big]\,d\lambda\\
-\int_0^\infty \frac{\kappa}{\Delta A^2}\Big[&6\tfrac{9}{16}\mathbf{A}^3\alpha^2+1\tfrac{7}{8}\mathbf{A}^2\mathbf{B}\alpha\beta+1\tfrac{7}{8}\mathbf{A}^2\mathbf{C}\alpha\gamma+\tfrac{9}{16}\mathbf{AB}^2\beta^2+\tfrac{9}{16}\mathbf{AC}^2\gamma^2+\tfrac{3}{8}\mathbf{ABC}\beta\gamma\Big]\,d\lambda\\
-\int_0^\infty \frac{1}{\Delta A^3}\Big[&6\tfrac{9}{16}\mathbf{A}^2L\alpha+\tfrac{3}{16}\mathbf{B}^2\left(M\alpha+2n\beta\right)+\tfrac{3}{16}\mathbf{C}^2\left(N\alpha+2m\gamma\right)\\
&+\tfrac{5}{8}\mathbf{AB}\left(L\beta+2n\alpha\right)+\tfrac{5}{8}\mathbf{AC}\left(L\gamma+2m\alpha\right)+\tfrac{1}{8}\mathbf{BC}\left(l\alpha+m\beta+n\gamma\right)\Big]\,d\lambda\\
+\int_0^\infty \frac{1}{\Delta A^2}\Big[&1\tfrac{5}{16}\mathbf{A}^2\left(L\kappa+2\alpha p\right)+\tfrac{3}{16}\mathbf{B}^2\left(M\kappa+2\beta q\right)+\tfrac{3}{16}\mathbf{C}^2\left(N\kappa+2\gamma r\right)\\
&+\tfrac{3}{8}\mathbf{AB}\left(n\kappa+\alpha q+\beta p\right)+\tfrac{3}{8}\mathbf{AC}\left(m\kappa+\gamma p+\alpha r\right)+\tfrac{1}{8}\mathbf{BC}\left(l\kappa+\gamma q+\beta r\right)\Big]\,d\lambda\\
+\int_0^\infty \frac{1}{\Delta A^3}\Big[&1\tfrac{1}{4}\mathbf{A}\mathfrak{L}+\tfrac{1}{4}\mathbf{B}\mathfrak{n}+\tfrac{1}{4}\mathbf{C}\mathfrak{m}\Big]\,d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)
\end{aligned}$$

$$\begin{aligned}
\delta_2 = \int_0^\infty \frac{1}{\Delta AB^2}\Big[&\kappa\left(3\tfrac{3}{4}\mathbf{A}^2\alpha\beta+2\tfrac{1}{4}\mathbf{AB}\beta^2+\tfrac{3}{4}\mathbf{AC}\beta\gamma\right)\\
&-\tfrac{3}{4}\mathbf{A}\left(n\kappa+\alpha q+\beta p\right)-\tfrac{3}{4}\mathbf{B}\left(M\kappa+2\beta q\right)-\tfrac{1}{4}\mathbf{C}\left(l\kappa+\gamma q+\beta r\right)\\
&+6\tfrac{9}{16}\mathbf{A}^3\alpha^2\beta+5\tfrac{5}{8}\mathbf{A}^2\mathbf{B}\alpha\beta^2+1\tfrac{7}{8}\mathbf{A}^2\mathbf{C}\alpha\beta\gamma\\
&+2\tfrac{13}{16}\mathbf{AB}^2\beta^3+\tfrac{9}{16}\mathbf{AC}^2\beta\gamma^2+\tfrac{9}{8}\mathbf{ABC}\beta^2\gamma\Big]\,d\lambda\\
-\int_0^\infty \frac{\kappa}{\Delta AB}\Big[&6\tfrac{9}{16}\mathbf{A}^3\alpha^2+1\tfrac{7}{8}\mathbf{A}^2\mathbf{B}\alpha\beta+1\tfrac{7}{8}\mathbf{A}^2\mathbf{C}\alpha\beta+\tfrac{9}{16}\mathbf{AB}^2\beta^2+\tfrac{9}{16}\mathbf{AC}^2\gamma^2+\tfrac{3}{8}\mathbf{ABC}\beta\gamma\Big]\,d\lambda\\
-\int_0^\infty \frac{1}{\Delta AB^2}\Big[&1\tfrac{5}{16}\mathbf{A}^2\left(L\beta+2n\alpha\right)+2\tfrac{13}{16}\mathbf{B}^2M\beta+\tfrac{3}{16}\mathbf{C}^2\left(N\beta+2l\gamma\right)\\
&+1\tfrac{1}{8}\mathbf{AB}\left(M\alpha+2n\beta\right)+\tfrac{3}{8}\mathbf{AC}\left(l\alpha+m\beta+n\gamma\right)+\tfrac{3}{8}\mathbf{BC}\left(M\gamma+2l\beta\right)\Big]\,d\lambda\\
+\int_0^\infty \frac{1}{\Delta AB}\Big[&1\tfrac{5}{16}\mathbf{A}^2\left(L\kappa+2\alpha p\right)+\tfrac{3}{16}\mathbf{B}^2\left(M\kappa+2\beta q\right)+\tfrac{3}{16}\mathbf{C}^2\left(N\kappa+2\gamma r\right)\\
&+\tfrac{3}{8}\mathbf{AB}\left(n\kappa+\alpha q+\beta p\right)+\tfrac{3}{8}\mathbf{AC}\left(m\kappa+\gamma p+\alpha r\right)+\tfrac{1}{8}\mathbf{BC}\left(l\kappa+\gamma q+\beta r\right)\Big]\,d\lambda\\
+\int_0^\infty \frac{1}{\Delta AB^2}\Big[&\tfrac{3}{4}\mathbf{A}\mathfrak{n}+\tfrac{3}{4}\mathbf{B}\mathfrak{M}+\tfrac{1}{4}\mathbf{C}\mathfrak{l}\Big]\,d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)
\end{aligned}$$

The value of δ_3 can be written down from symmetry; that of δ_4, which is of the same general type, will not be required in the present investigation.

13. This completes the solution of the particular potential problem which we have had in hand. It might naturally be feared that some mistake might have been made either in principle or in detail, and it must be remembered that even one mistake might invalidate the answer to the whole problem. I have, therefore, both here and elsewhere, taken the utmost care to check the accuracy of my work in every way. The following will, I think, show that no error need be feared in the solution which has just been obtained.

The value of V_i, if obtained accurately, ought to satisfy $\nabla^2 V_i = -4\pi\rho$, and the term $-4\pi\rho$ must come entirely from the terms independent of e in V_i. Thus the terms in V_i which are multiplied by e, e^2 and e^3 ought separately to be spherical harmonics.

It was verified in the previous paper that the terms in e and e^2 were in actual fact of this form. The terms in e^3 will be harmonic if the quantity on the right of equation (57) is harmonic, and the conditions for this are expressed by the equations

$$10\mathfrak{C}_{11}+\mathfrak{C}_{12}+\mathfrak{C}_{13} = 0, \qquad (68)$$

$$6\mathfrak{C}_{22}+3\mathfrak{C}_{12}+\mathfrak{C}_{23} = 0, \qquad (69)$$

$$6\mathfrak{C}_{33}+3\mathfrak{C}_{13}+\mathfrak{C}_{23} = 0, \qquad (70)$$

$$3\mathfrak{D}_1+\mathfrak{D}_2+\mathfrak{D}_3 = 0. \qquad (71)$$

I have inserted the values just obtained for $\mathfrak{C}_{11}$, $\mathfrak{C}_{12}$, ... $\mathfrak{D}_1$, ... in these equations, and have verified that they are all satisfied. (The necessary transformations of the various integrals are tedious, but involve no special difficulties.)

It follows that the solution we have obtained gives accurately the potential of *some* solid of uniform density ρ. By the method explained in the earlier sections of the previous paper, it is easy to work the problem backwards and to verify that the equation of the boundary of the solid in question is obtained by putting $\lambda = 0$ in the equation $f+\phi = 0$. Thus we verify that our solution gives accurately the potential of the solid of boundary

$$f+eP+e^2Q+e^3R = 0. \qquad (72)$$

Conditions that Pear-shaped Figure shall be one of Equilibrium for a Rotating Liquid.

14. In order that the figure determined by equation (72) shall be one of equilibrium for a rotating liquid, the potential at the boundary plus $\frac{1}{2}w^2(x^2+y^2)$ must, as in equation (98) of the former paper, be identical with

$$-\pi\rho abc\theta\,\{f+eP+e^2Q+e^3R\}+\text{a constant}.$$

Let us limit ourselves to the terms in e^3. Using the value of V_b given by equation (56) and that of R assumed in equation (52), we find that we must have

$$-\pi\rho abce^3x\left(\mathfrak{c}_{11}x^4+\mathfrak{c}_{22}y^4+\mathfrak{c}_{33}z^4+\mathfrak{c}_{12}x^2y^2+\mathfrak{c}_{31}z^2x^2+\mathfrak{c}_{23}y^2z^2+\mathfrak{d}_1x^2+\mathfrak{d}_2y^2+\mathfrak{d}_3z^2+\mathfrak{d}_4\right)$$

$$= -\pi\rho abc\theta e^3\tfrac{1}{4}\frac{x}{a^2}\left(\mathfrak{L}\frac{x^4}{a^8}+\mathfrak{M}\frac{y^4}{b^8}+\mathfrak{N}\frac{z^4}{c^8}+2\mathfrak{l}\frac{y^2z^2}{b^4c^4}+2\mathfrak{m}\frac{z^2x^2}{c^4a^4}\right.$$
$$\left.+2\mathfrak{n}\frac{x^2y^2}{a^4b^4}+2\left(\mathfrak{p}\frac{x^2}{a^4}+\mathfrak{q}\frac{y^2}{b^4}+\mathfrak{r}\frac{z^2}{c^4}\right)+\mathfrak{s}\right).$$

On equating coefficients we obtain

$$\mathfrak{c}_{11}=\tfrac{1}{4}\frac{\theta}{a^{10}}\mathfrak{L};\qquad \mathfrak{c}_{22}=\tfrac{1}{4}\frac{\theta}{a^2b^8}\mathfrak{M};\qquad \mathfrak{c}_{33}=\tfrac{1}{4}\frac{\theta}{a^2c^8}\mathfrak{N}, \quad (73)$$

$$\mathfrak{c}_{23}=\tfrac{1}{2}\frac{\theta}{a^2b^4c^4}\mathfrak{l};\qquad \mathfrak{c}_{31}=\tfrac{1}{2}\frac{\theta}{a^6c^4}\mathfrak{m};\qquad \mathfrak{c}_{12}=\tfrac{1}{2}\frac{\theta}{a^6b^4}\mathfrak{n}, \quad (74)$$

$$\mathfrak{d}_1=\tfrac{1}{2}\frac{\theta}{a^6}\mathfrak{p};\qquad \mathfrak{d}_2=\tfrac{1}{2}\frac{\theta}{a^2b^4}\mathfrak{q};\qquad \mathfrak{d}_3=\tfrac{1}{2}\frac{\theta}{a^2c^4}\mathfrak{r}, \quad (75)$$

$$\mathfrak{d}_4=\tfrac{1}{4}\frac{\theta}{a^2}\mathfrak{s}. \quad (76)$$

These equations, in addition to those found for the first- and second-order terms in the previous paper, express the condition that the third-order figure (72) shall be a possible figure of equilibrium.

15. On substituting the values of $\mathfrak{c}_{11}, \mathfrak{c}_{12} \ldots \mathfrak{c}_{33}$ which have been obtained in § 12 into the six equations (73) and (74), we obtain a system of six equations from which it is possible to determine the six unknowns $\mathfrak{L}, \mathfrak{M}, \mathfrak{N}, \mathfrak{l}, \mathfrak{m}, \mathfrak{n}$. The solution is actually effected in § 17 below.

If we substitute the three values of $\mathfrak{d}_1$, $\mathfrak{d}_2$ and $\mathfrak{d}_3$ obtained in § 12 (equations (62)–(64)) into the three equations (75), we obtain three equations which can be written in the form :—

$$\mathfrak{p}\left(\tfrac{1}{2}\frac{\theta}{a^6}+\tfrac{3}{4}\int_0^\infty\frac{\mathrm{A}d\lambda}{\Delta\mathrm{A}^2}-\tfrac{1}{2}\int_0^\infty\frac{d\lambda}{\Delta\mathrm{A}^3}\right)+\tfrac{1}{4}\mathfrak{q}\int_0^\infty\frac{\mathrm{B}d\lambda}{\Delta\mathrm{A}^2}+\tfrac{1}{4}\mathfrak{r}\int_0^\infty\frac{\mathrm{C}d\lambda}{\Delta\mathrm{A}^2}=\delta_1-2\int_0^\infty\frac{\mathrm{G}}{\Delta\mathrm{A}^2}d\lambda, \quad (77)$$

$$\tfrac{3}{4}\mathfrak{p}\int_0^\infty\frac{\mathrm{A}d\lambda}{\Delta\mathrm{AB}}+\mathfrak{q}\left(\tfrac{1}{2}\frac{\theta}{a^2b^4}+\tfrac{1}{4}\int_0^\infty\frac{\mathrm{B}d\lambda}{\Delta\mathrm{AB}}-\tfrac{1}{2}\int_0^\infty\frac{d\lambda}{\Delta\mathrm{AB}^2}\right)+\tfrac{1}{4}\mathfrak{r}\int_0^\infty\frac{\mathrm{C}d\lambda}{\Delta\mathrm{AB}}=\delta_2-2\int_0^\infty\frac{\mathrm{G}}{\Delta\mathrm{AB}}d\lambda, \quad (78)$$

$$\tfrac{3}{4}\mathfrak{p}\int_0^\infty\frac{\mathrm{A}d\lambda}{\Delta\mathrm{AC}}+\tfrac{1}{4}\mathfrak{q}\int_0^\infty\frac{\mathrm{B}d\lambda}{\Delta\mathrm{AC}}+\mathfrak{r}\left(\tfrac{1}{2}\frac{\theta}{a^2c^4}+\tfrac{1}{4}\int_0^\infty\frac{\mathrm{C}d\lambda}{\Delta\mathrm{AC}}-\tfrac{1}{2}\int_0^\infty\frac{d\lambda}{\Delta\mathrm{AC}^2}\right)=\delta_3-2\int_0^\infty\frac{\mathrm{G}}{\Delta\mathrm{AC}}d\lambda. \quad (79)$$

It will be seen that $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ do not occur on the right-hand sides of these equations.

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These equations are not, as might at first be thought, a system of three simple equations determining $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$. They will be found to be of the type known as "porismatic"*; that is to say, equations which are inconsistent unless the coefficients satisfy a certain relation, and such that, when this relation is satisfied, the equations have an infinite number of solutions.

Let us, for brevity, write the equations in the form :—

$$\left.\begin{aligned} k_1\mathfrak{p} + k'_1\mathfrak{q} + k''_1\mathfrak{r} &= \mathfrak{K}_1 \\ k_2\mathfrak{p} + k'_2\mathfrak{q} + k''_2\mathfrak{r} &= \mathfrak{K}_2 \\ k_3\mathfrak{p} + k'_3\mathfrak{q} + k''_3\mathfrak{r} &= \mathfrak{K}_3 \end{aligned}\right\} \quad . \quad . \quad . \quad (80)$$

Let us use also the abbreviated notation of the previous paper (p. 50), such that

$$c_1 = \int_0^\infty \frac{\lambda d\lambda}{\Delta\,\mathrm{ABC}}, \qquad c_2 = \int_0^\infty \frac{\lambda d\lambda}{\Delta\,\mathrm{A^2C}}, \qquad c_3 = \int_0^\infty \frac{\lambda d\lambda}{\Delta\,\mathrm{A^2B}}.$$

Then, by simple transformations of the integrals, we obtain

$$\left.\begin{aligned} k_1 &= \frac{1}{4a^2}\left(\frac{2\theta}{a^4} - c_2 - c_3\right), & k'_1 &= \frac{1}{4b^2}c_3, & k''_1 &= \frac{1}{4c^2}c_2, \\ k_2 &= \frac{3}{4a^2}c_3, & k'_2 &= \frac{1}{4b^2}\left(\frac{2\theta}{a^2b^2} - 3c_3 - c_1\right), & k''_2 &= \frac{1}{4c^2}c_1, \\ k_3 &= \frac{3}{4a^2}c_2, & k'_3 &= \frac{1}{4b^2}c_1, & k''_3 &= \frac{1}{4c^2}\left(\frac{2\theta}{a^2c^2} - 3c_2 - c_1\right), \end{aligned}\right\} \quad . \quad . \quad . \quad (81)$$

With these values of the coefficients, it will be found that equations (71)–(73) of the previous paper reduce to

$$\left.\begin{aligned} 2k_1\alpha + 2k'_1\beta + 2k''_1\gamma &= 0 \\ 2k_2\alpha + 2k'_2\beta + 2k''_2\gamma &= 0 \\ 2k_3\alpha + 2k'_3\beta + 2k''_3\gamma &= 0 \end{aligned}\right\}, \quad . \quad . \quad . \quad (82)$$

so that

$$\mathfrak{p} = 2\alpha, \qquad \mathfrak{q} = 2\beta, \qquad \mathfrak{r} = 2\gamma, \quad . \quad . \quad . \quad (83)$$

is a solution of equations (80), when $\mathfrak{K}_1 = \mathfrak{K}_2 = \mathfrak{K}_3 = 0$.

Without this detailed inspection of the equations, it could have been foreseen that this would necessarily be the case. For our general solution

$$\phi = e\mathrm{P} + e^2\mathrm{Q} + e^3\mathrm{R}, \quad . \quad . \quad . \quad (84)$$

* See 'HOBSON'S Plane Trigonometry,' § 73, or WOLSTENHOLME, 'Proc. London Math. Soc.,' vol. 4.

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must cover all possible figures of equilibrium as far as the third order. One such figure is, however, known to be

$$\phi = e \, . \, 0 + e^2 \, . \, 0 + e^3 \mathrm{P}, \quad \ldots \ldots \ldots \quad (85)$$

and the corresponding solution for $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ is that expressed by equations (83).

From equations (82), it follows at once that we must have

$$\begin{vmatrix} k_1, & k'_1, & k''_1 \\ k_2, & k'_2, & k''_2 \\ k_3, & k'_3, & k''_3 \end{vmatrix} = 0. \quad \ldots \ldots \ldots \quad (86)$$

Indeed, it is now clear that this is precisely the equation which determined the existence of the point of bifurcation on the series of Jacobian ellipsoids.*

If this were the only relation between the coefficients in equations (80), these equations could have no solution other than $\mathfrak{p} = \mathfrak{q} = \mathfrak{r} = \infty$. Let us, however, multiply the three equations (80) by the three minors of k''_1, k''_2, k''_3 in the determinant of equation (86), and add. We obtain a relation of the type,

$$\mathfrak{p} \, . \, 0 + \mathfrak{q} \, . \, 0 + \mathfrak{r} \, . \, 0 = \begin{vmatrix} k_1, & k'_1, & \mathfrak{K}_1 \\ k_2, & k'_2, & \mathfrak{K}_2 \\ k_3, & k'_3, & \mathfrak{K}_3 \end{vmatrix}, \quad \ldots \ldots \quad (87)$$

and it is clear that these equations can now have a solution in which $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ are not all infinite if we have

$$\begin{vmatrix} k_1, & k'_1, & \mathfrak{K}_1 \\ k_2, & k'_2, & \mathfrak{K}_2 \\ k_3, & k'_3, & \mathfrak{K}_3 \end{vmatrix} = 0. \quad \ldots \ldots \ldots \quad (88)$$

It is only when this relation is satisfied that it is possible to continue our linear series of equilibrium configurations beyond the second order terms. When it is not satisfied, our third order solution (84) lapses back into the solution (85), namely $\phi = e^3 \mathrm{P}$, which is virtually the first order solution with e^3 replacing e as parameter.

16. The relation (88) can be expressed in a much simpler form.

Independently of the values of $\mathfrak{p}$, $\mathfrak{q}$ and $\mathfrak{r}$, we have already seen (equation (71)) that we must have

$$3\mathfrak{D}_1 + \mathfrak{D}_2 + \mathfrak{D}_3 = 0. \quad \ldots \ldots \ldots \quad (89)$$

* As to the relation of this equation to the general theory, see § 36 of my previous paper.

On substituting the values for $\mathfrak{d}_1$, $\mathfrak{d}_2$ and $\mathfrak{d}_3$ given by equations (62)–(64), and equating coefficients of $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ in equation (89) to zero, we obtain

$$3\mathfrak{K}_1 + \mathfrak{K}_2 + \mathfrak{K}_3 = 0.$$

$$3k_1 + k_2 + k_3 = \tfrac{3}{2}\frac{\theta}{a^6}.$$

$$3k'_1 + k'_2 + k'_3 = \tfrac{1}{2}\frac{\theta}{a^2b^4}.$$

$$3k''_1 + k''_2 + k''_3 = \tfrac{1}{2}\frac{\theta}{a^2c^4}.$$

With the help of these relations equation (88) reduces quite simply to

$$\mathfrak{K}_1\left(\frac{3}{a^4}k'_2 - \frac{1}{b^4}k_2\right) = \mathfrak{K}_2\left(\frac{3}{a^4}k'_1 - \frac{1}{b^4}k_1\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

In this equation the coefficients of $\mathfrak{K}_1$ and $\mathfrak{K}_2$ depend only on a, b, c the semi-axes of the Jacobian ellipsoid, and so are fully known. The quantities $\mathfrak{K}_1$, $\mathfrak{K}_2$ however depend on the second-order coefficients L, M, N, ... p, q, r, s. These were calculated in the previous paper, but p, q, r, s could not be fully determined, since they were found to depend on a quantity n'', which measured the change in angular velocity This it was found impossible to evaluate so long as the investigation was confined to second-order terms. It now appears that equation (90) is in effect an equation determining n''. The equation is linear in n'', so that it gives only one value for n''. When n'' has this value we are on the true linear series, but if n'' has any other value our solution, when we try to extend it to third-order terms, degenerates into a solution of the type of (85), with which no progress can be made. Our plan, then, is to evaluate the terms which occur in equation (90) and so obtain the value of n''. On inserting this into the values of p, q, r, s which were obtained in the previous paper, we complete the solution as far as the second-order terms, and can then proceed to the stability criterion.

Numerical Computations.

17. It is at once apparent that the evaluation of $\mathfrak{c}_{11}$, $\mathfrak{c}_{12}$, ... $\mathfrak{d}_1$, $\mathfrak{d}_2$, ..., given by equations (58)–(67), can be made to depend on integrals of the same type as occurred in the previous paper, namely integrals defined by

$$J_{ABC\ldots} = \int_0^\infty \frac{d\lambda}{\Delta ABC\ldots}.$$

In addition to the integrals which were given in the previous paper,* I have calculated the following values of additional integrals needed for the computation of third-order terms.

$J_{B^5} = 0{\cdot}9637364$,	$J_{B^4C} = 1{\cdot}392786$,	$J_{B^3C^2} = 2{\cdot}024097$,
$J_{B^2C^3} = 2{\cdot}948174$,	$J_{BC^4} = 4{\cdot}316064$,	$J_{C^5} = 6{\cdot}336915$,

$J_{A^6} = 0{\cdot}0001716853$,	$J_{A^5B} = 0{\cdot}0006851396$,	$J_{A^5C} = 0{\cdot}000942028$,
$J_{A^4B^2} = 0{\cdot}002877084$,	$J_{A^4BC} = 0{\cdot}004027127$,	$J_{A^4C^2} = 0{\cdot}005675881$,
$J_{A^3B^3} = 0{\cdot}01256355$,	$J_{A^3B^2C} = 0{\cdot}01783828$,	$J_{A^3BC^2} = 0{\cdot}02547622$,
$J_{A^3C^3} = 0{\cdot}0365761$,	$J_{A^2B^4} = 0{\cdot}0565289$,	$J_{A^2B^3C} = 0{\cdot}0811833$,
$J_{A^2B^2C^2} = 0{\cdot}1172029$,	$J_{A^2BC^3} = 0{\cdot}1698776$,	$J_{A^2C^4} = 0{\cdot}2473229$,
$J_{AB^5} = 0{\cdot}260331$,	$J_{AB^4C} = 0{\cdot}377266$,	$J_{AB^3C^2} = 0{\cdot}549771$,
$J_{AB^2C^3} = 0{\cdot}802462$,	$J_{ABC^4} = 1{\cdot}177387$,	$J_{AC^5} = 1{\cdot}731910$,
$J_{B^6} = 1{\cdot}221051$,	$J_{B^5C} = 1{\cdot}781565$,	$J_{B^4C^2} = 2{\cdot}621427$,
$J_{B^3C^3} = 3{\cdot}837093$,	$J_{B^2C^4} = 5{\cdot}679960$,	$J_{BC^5} = 8{\cdot}391280$,
	$J_{C^6} = 12{\cdot}448855$,	

$J_{A^7} = 0{\cdot}0000434660$,	$J_{A^6B} = 0{\cdot}000177533$,	$J_{A^6C} = 0{\cdot}000245881$,
$J_{A^5B^2} = 0{\cdot}000757892$,	$J_{A^5BC} = 0{\cdot}00106671$,	$J_{A^5C^2} = 0{\cdot}00151097$,
$J_{A^4B^3} = 0{\cdot}00334922$,	$J_{A^4B^2C} = 0{\cdot}00477536$,	$J_{A^4BC^2} = 0{\cdot}00684621$,
$J_{A^4C^3} = 0{\cdot}00986287$,	$J_{A^3B^4} = 0{\cdot}0152016$,	$J_{A^3B^3C} = 0{\cdot}0219023$,
$J_{A^3B^2C^2} = 0{\cdot}0317157$,	$J_{A^3BC^3} = 0{\cdot}0460906$,	$J_{A^3C^4} = 0{\cdot}0672671$,
$J_{A^2B^5} = 0{\cdot}0704673$,	$J_{A^2B^4C} = 0{\cdot}102374$,	$J_{A^2B^3C^2} = 0{\cdot}149566$,
$J_{A^2B^2C^3} = 0{\cdot}218724$,	$J_{A^2BC^4} = 0{\cdot}321581$,	$J_{A^2C^5} = 0{\cdot}473857$,
$J_{AB^6} = 0{\cdot}332181$,	$J_{AB^5C} = 0{\cdot}485553$,	$J_{AB^4C^2} = 0{\cdot}716300$,
$J_{AB^3C^3} = 1{\cdot}04926$,	$J_{AB^2C^4} = 1{\cdot}55682$,	$J_{ABC^5} = 2{\cdot}30256$,
	$J_{AC^6} = 3{\cdot}42068$.	

The first task is the evaluation of $\mathfrak{c}_{11}, \mathfrak{c}_{12}, \ldots, \mathfrak{c}_{33}$. These quantities are connected by the three relations (68)–(70), so that only three of the six quantities need have been calculated. I have, however, calculated four coefficients, namely, $\mathfrak{c}_{12}$, $\mathfrak{c}_{13}$, $\mathfrak{c}_{22}$ and $\mathfrak{c}_{33}$ quite independently, the object being to obtain a check on the accuracy of the computations. In this way a check is obtained at every step of the computations.

* The following are true values of integrals which were incorrectly given in the previous paper:—

$$J_{BCC} = 1{\cdot}3419631, \qquad J_{AABCC} = 0{\cdot}09510324, \qquad I_{CCCC} = 0{\cdot}4781180.$$

In each case the error was one of printing only, and did not enter into the computations. The whole table given in the previous paper was recomputed before extending it for purposes of the present paper.

The values obtained were

$$\mathfrak{c}_{12} = -0{\cdot}0002799\mathfrak{L} - 0{\cdot}0093206\mathfrak{M} + 0{\cdot}0103775\mathfrak{N} \\ -0{\cdot}0045815\mathfrak{l} - 0{\cdot}0016151\mathfrak{m} + 0{\cdot}0040268\mathfrak{n} + 0{\cdot}0042388, \quad . \quad . \quad (91)$$

$$\mathfrak{c}_{13} = -0{\cdot}0003503\mathfrak{L} + 0{\cdot}0035301\mathfrak{M} - 0{\cdot}0295936\mathfrak{N} \\ -0{\cdot}0024284\mathfrak{l} + 0{\cdot}0074994\mathfrak{m} - 0{\cdot}0010296\mathfrak{n} + 0{\cdot}0044353, \quad . \quad . \quad (92)$$

$$\mathfrak{c}_{22} = 0{\cdot}0001010\mathfrak{L} + 0{\cdot}0126532\mathfrak{M} + 0{\cdot}0153304\mathfrak{N} \\ -0{\cdot}0250819\mathfrak{l} + 0{\cdot}0010344\mathfrak{m} - 0{\cdot}0017407\mathfrak{n} - 0{\cdot}0015791, \quad . \quad . \quad (93)$$

$$\mathfrak{c}_{33} = 0{\cdot}0001361\mathfrak{L} + 0{\cdot}0062287\mathfrak{M} + 0{\cdot}0353141\mathfrak{N} \\ -0{\cdot}0261575\mathfrak{l} - 0{\cdot}0035230\mathfrak{m} + 0{\cdot}0007874\mathfrak{n} - 0{\cdot}0016798. \quad . \quad . \quad (94)$$

Two values of $\mathfrak{c}_{23}$ can now be deduced from equations (69) and (70) respectively. These are found to be

$$\mathfrak{c}_{23} = \begin{array}{l} +0{\cdot}0002337 \\ +0{\cdot}0002343 \end{array} \mathfrak{L} \begin{array}{l} -0{\cdot}047957 \\ -0{\cdot}047962 \end{array} \mathfrak{M} \begin{array}{l} -0{\cdot}123015 \\ -0{\cdot}123004 \end{array} \mathfrak{N} \\ \begin{array}{l} +0{\cdot}16424 \\ +0{\cdot}16420 \end{array} \mathfrak{l} \begin{array}{l} -0{\cdot}001361 \\ -0{\cdot}001360 \end{array} \mathfrak{m} \begin{array}{l} -0{\cdot}001636 \\ -0{\cdot}001636 \end{array} \mathfrak{n} \begin{array}{l} -0{\cdot}003242 \\ -0{\cdot}003227. \end{array}$$

The agreement of these values provides a check on the computations of the coefficients $\mathfrak{c}_{12}$, ..., and of the integrals from which they have been calculated.

In virtue of relations (73), equations (68)–(70) become

$$5\frac{\mathfrak{L}}{a^4} + \frac{\mathfrak{n}}{b^4} + \frac{\mathfrak{m}}{c^4} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

$$3\frac{\mathfrak{M}}{b^4} + \frac{3\mathfrak{n}}{a^4} + \frac{\mathfrak{l}}{c^4} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

$$3\frac{\mathfrak{N}}{c^4} + \frac{3\mathfrak{m}}{a^4} + \frac{\mathfrak{l}}{b^4} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

while relations (73) and (74), of which only three are now independent, may be represented by

$$\mathfrak{c}_{12} - \tfrac{1}{2}\frac{\theta}{a^6 b^4}\mathfrak{n} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

$$\mathfrak{c}_{13} - \tfrac{1}{2}\frac{\theta}{a^6 c^4}\mathfrak{m} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

$$\mathfrak{c}_{22} - \tfrac{1}{4}\frac{\theta}{a^2 b^8}\mathfrak{M} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

On substituting for $\mathfrak{c}_{12}$, $\mathfrak{c}_{13}$, and $\mathfrak{c}_{22}$ from equations (91)–(98), these become six linear equations for $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{l}$, $\mathfrak{m}$, and $\mathfrak{n}$.

The solution of these equations is found to be

$$\mathfrak{L} = -12{\cdot}6275, \qquad \mathfrak{M} = -0{\cdot}0307056, \qquad \mathfrak{N} = -0{\cdot}0044636,$$
$$\mathfrak{l} = -\ 0{\cdot}0116194, \qquad \mathfrak{m} = 0{\cdot}42602, \qquad \mathfrak{n} = 1{\cdot}15365.$$

These values have been checked by insertion, not only in the six equations from which they were directly derived, but also in the remaining equations (73) and (74).

We now have all the material necessary for the evaluation of $\mathfrak{K}_1$, $\mathfrak{K}_2$ and $\mathfrak{K}_3$, which, it will be remembered, are the right-hand members of equations (77)–(79) respectively. These have been computed independently, and I found

$$\mathfrak{K}_1 = -0{\cdot}0016803+0{\cdot}228894n'',$$
$$\mathfrak{K}_2 = +0{\cdot}0026780-0{\cdot}301110n'',$$
$$\mathfrak{K}_3 = +0{\cdot}0023739-0{\cdot}385644n''.$$

These values ought to satisfy (*cf.* § 16)

$$3\mathfrak{K}_1+\mathfrak{K}_2+\mathfrak{K}_3 = 0\,,$$

in place of which I find

$$3\mathfrak{K}_1+\mathfrak{K}_2+\mathfrak{K}_3 = 0{\cdot}000011-0{\cdot}00007n'',$$

but the error is no greater than might reasonably be expected in view of the very large number of operations in each computation.*

The coefficients in equation (90) are found to be

$$\frac{3}{a^4}k'_2-\frac{1}{b^4}k_2 = 0{\cdot}0058753, \qquad \frac{3}{a^4}k'_1-\frac{1}{b^4}k_1 = 0{\cdot}00024949,$$

so that the equation itself becomes

$$5{\cdot}8753\,(-0{\cdot}0016803+0{\cdot}228894n'') = 0{\cdot}24949\,(0{\cdot}0026780-0{\cdot}301110n''),$$

and the solution is found to be

$$n'' = 0{\cdot}007423.$$

Completion of Second Order Solution.

18. On substituting the value just obtained for n'' into the values for p, q, r, s, found in the previous paper (§ 34), I find,

$$p = 3{\cdot}124954,$$
$$q = -0{\cdot}103164,$$
$$r = -0{\cdot}015236,$$
$$s = -0{\cdot}256962,$$

thus completing the figure to the second-order terms.

* Each of the quantities $\mathfrak{K}_1$, $\mathfrak{K}_2$, $\mathfrak{K}_3$ has been computed by expressing it as a sum of integrals of the type tabulated on p. 21. The first term in $\mathfrak{K}_1$, namely $13\frac{1}{8}\kappa\alpha^2a^{-4}J_{AAA}$, may be thought of as a typical term. Each of the quantities $\mathfrak{K}_1$, $\mathfrak{K}_2$, $\mathfrak{K}_3$ consisted of 326 such terms, so that $3\mathfrak{K}_1+\mathfrak{K}_2+\mathfrak{K}_3$ is a sum of 978 such terms, each of which, it must be remembered, is evaluated by a fairly lengthy series of computations.

The corresponding rotation is given by

$$\frac{\omega^2}{2\pi\rho} = n + e^2 n'' = 0{\cdot}14200 + 0{\cdot}007423e^2.$$

We notice at once that ω increases as we pass along the pear-shaped series, whereas in Sir G. DARWIN's solution ω was found to decrease. Thus the present solution diverges in essentials from that of DARWIN.

On the other hand the present solution is similar to that for rotating cylinders* in which the rotation was also found to increase as we passed along the pear-shaped series, and as we shall now see, the increase is at a very similar rate.

Our present figure, as far as the first order of small quantities, is

$$\frac{x^2}{3{\cdot}55} + \frac{y^2}{0{\cdot}664} + \frac{z^2}{0{\cdot}424} + e\,(-0{\cdot}079x^3 + 0{\cdot}127xy^2 + 0{\cdot}106xz^2 + 0{\cdot}142x) = 1,$$

while the cylindrical figure, on replacing the parameter $10^{3/2}\theta$ used in the second half of the two-dimensional investigation by e, was found to be

$$\frac{x^2}{5} + \frac{y^2}{0{\cdot}555} + e\,(-0{\cdot}063x^3 + 0{\cdot}190xy^2 + 0{\cdot}211x) = 1.$$

A comparison of these two figures shows that the two e's may be taken to be very approximately the same. As regards angular velocity, the value in the present three-dimensional problem has been found to be

$$\frac{\omega^2}{2\pi\rho} = n + n''e^2 = 0{\cdot}14200\,(1 + 0{\cdot}05227e^2),$$

while that in the two-dimensional problems was†

$$\frac{\omega^2}{2\pi\rho} = \frac{3}{8} + \frac{8{\cdot}625}{448}e^2 = 0{\cdot}3750\,(1 + 0{\cdot}0513e^2).$$

Thus, in so far as it is possible to compare the three-dimensional problem with its two-dimensional analogue, we may say that the two rotations are in very close agreement.

Calculation of the Moment of Inertia.

19. We know that the pear-shaped figure will be stable or unstable according as the angular momentum increases or decreases as we pass from the critical Jacobian ellipsoid along the series of pear-shaped figures.

* "On the Equilibrium of Rotating Liquid Cylinders," 'Phil. Trans.,' A, vol. 200 (1902), p. 67.

† I have recently repeated the calculations of the two-dimensional problem as far as the third order of small quantities, including the evaluation of ω^2, and find that the numbers given originally for the figure and angular velocity are absolutely correct. See, however, § 21 below.

The moment of inertia of the pear-shaped figure about the axis of rotation, say Mk^2, will be given by

$$Mk^2 = \iiint \rho\,(x^2+y^2)\,dx\,dy\,dz$$

where the integral is taken throughout the volume of the pear-shaped figure. We have, by our choice of the coefficient s, ensured that the volume of the pear-shaped figure shall remain always equal to that of the original ellipsoid, so that we have

$$M = \frac{4}{3}\pi\rho abc,$$

and therefore

$$k^2 = \frac{3}{4\pi}\iiint (x^2+y^2)\,\frac{dx\,dy\,dz}{abc} \quad \ldots \quad (101)$$

Let us transform to co-ordinates x', y', z', given by

$$x' = \frac{x}{a}, \qquad y' = \frac{y}{b}, \qquad z' = \frac{z}{c} \quad \ldots \quad (102)$$

so that the critical Jacobian ellipsoid is reduced to a sphere of unit radius, and the pear-shaped figure is reduced to a distorted sphere. With this transformation, equation (101) becomes

$$k^2 = \frac{3}{4\pi}\iiint (a^2x'^2+b^2y'^2)\,du'\,dy'\,dz', \quad \ldots \quad (103)$$

where the integral is taken throughout the figure bounded by the surface

$$x'^2+y'^2+z'^2-1+e\frac{x'}{a}\left(\alpha\frac{x'^2}{a^2}+\beta\frac{y'^2}{b^2}+\gamma\frac{z'^2}{c^2}+\kappa\right)$$
$$+\tfrac{1}{4}e^2\left[\frac{Lx'^4}{a^4}+\frac{My'^4}{b^4}+\ldots+s\right]=0. \quad \ldots \quad (104)$$

Let r^2 be written for $x'^2+y'^2+z'^2$, and let us further put

$$x' = r\mathrm{x}, \qquad y' = r\mathrm{y}, \qquad z' = r\mathrm{z},$$

so that x, y, z, are co-ordinates on a sphere of unit radius. Equation (104) becomes

$$r^2-1+er^3\frac{\mathrm{x}}{a}\left(\alpha\frac{\mathrm{x}^2}{a^2}+\beta\frac{\mathrm{y}^2}{b^2}+\gamma\frac{\mathrm{z}^2}{c^2}\right)+er\frac{\mathrm{x}\kappa}{a}$$
$$+\tfrac{1}{4}e^2\left[\frac{L\mathrm{x}^4r^4}{a^4}+\frac{M\mathrm{y}^4r^4}{b^4}+\ldots+s\right]=0. \quad \ldots \quad (105)$$

Let us suppose that r, the radius vector to the boundary of this distorted unit sphere is given by

$$r = 1+e\mathrm{f}+e^2\mathrm{g}+e^3\mathrm{h}+\ldots.$$

On substituting this value for r into equation (105) and equating the coefficients of e and e^2, we obtain

$$2\mathrm{f}+\frac{\mathrm{x}}{a}\left[\alpha\frac{\mathrm{x}^2}{a^2}+\beta\frac{\mathrm{y}^2}{b^2}+\gamma\frac{\mathrm{z}^2}{c^2}+\kappa\right]=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

$$\mathrm{f}^2+2\mathrm{g}+\mathrm{f}\frac{\mathrm{x}}{a}\left[3\left(\alpha\frac{\mathrm{x}^2}{a^2}+\beta\frac{\mathrm{y}^2}{b^2}+\gamma\frac{\mathrm{z}^2}{c^2}\right)+\kappa\right]+\tfrac{1}{4}\left[\frac{\mathrm{L}\mathrm{x}^4}{a^4}+\frac{\mathrm{M}\mathrm{y}^4}{b^4}+\ldots+s\right]=0. \quad . \quad (107)$$

If $d\Omega$ is an element of solid angle, equation (103) may be written in the form

$$k^2=\frac{3}{4\pi}\iiint(a^2\mathrm{x}^2+b^2\mathrm{y}^2)\,r^4\,dr\,d\Omega,$$

$$=\frac{3}{20\pi}\iint(a^2\mathrm{x}^2+b^2\mathrm{y}^2)\,(1+5e\mathrm{f}+5e^2\mathrm{g}+10e^2\mathrm{f}^2)d\Omega.$$

Hence we find that k^2 may be written in the form $k_0^{\ 2}+\Delta k^2$, where, as far as e^2,

$$\Delta k^2=\frac{3}{4\pi}\iint(a^2\mathrm{x}^2+b^2\mathrm{y}^2)\,(\mathrm{g}+2\mathrm{f}^2)\,d\Omega.$$

The integral is here taken over the sphere of unit radius, and so can be easily evaluated. Carrying out the necessary computations, I find

$$k_0^{\ 2}=\tfrac{1}{5}\,(a^2+b^2)=0{\cdot}844105.$$

$$\Delta k^2=-0{\cdot}079156e^2.$$

Thus the moment of inertia is given by

$$\mathrm{M}k^2=0{\cdot}844105\mathrm{M}\,(1-0{\cdot}09378e^2)$$

20. This again differs from Sir G. DARWIN's result, in which it will be remembered it was found that the moment of inertia of the pear increased as e^2 increased. But we shall now see that the difference between the two results agrees exactly with what was to be expected from the different values of n'' used in our two solutions.

Sir G. DARWIN chose his parameter e (which I shall denote by e_{D}) in such a way that the longest radius vector of the pear-shaped figure was $a\,(1+0{\cdot}1482e_{\mathrm{D}})$. In my solution, the longest radius vector is found to be $a\,(1+0{\cdot}1309e_{\mathrm{J}})$, where e_{J} denotes my parameter e. Hence I find as the relation between our parameters

$$e_{\mathrm{D}}=0{\cdot}8833e_{\mathrm{J}}.$$

DARWIN's rotation was given by

$$\frac{\omega^2}{2\pi\rho}=0{\cdot}14200(1-0{\cdot}1443066e_{\mathrm{D}}^{\ 2})=0{\cdot}14200\,(1-0{\cdot}11259e_{\mathrm{J}}^{\ 2}).$$

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The general solution, apart from special values for the rotation, is such that, in my notation,

$$\frac{\omega^2}{2\pi\rho} = n + e_J^2 n'' = 0{\cdot}14200\left(1 + \frac{n''}{0{\cdot}14200} e_J^2\right).$$

Hence I find that DARWIN's solution ought to coincide with the solution obtained in my previous paper on assigning to n'' a value n''_D given by

$$n''_D = -0{\cdot}015988.$$

The solution given in the present paper, which is believed to give a true figure of equilibrium, is derived from the general solution by assigning to n'' a value n''_J given by

$$n''_J = +0{\cdot}007423.$$

In my previous paper, it was shown that there was, so far as second-order terms only were concerned, a doubly infinite series of figures of equilibrium, and it was found that these could be defined in terms of two independent parameters, e and ζ, where ζ was the same thing as $e_J^2 n''$. It accordingly appears that any figure of DARWIN's series differs from my figure of equilibrium having the same value of e, through his ζ being different from mine. The excess of my ζ over his will be

$$\zeta_{J-D} = e_J^2 (n''_J - n''_D) = +0{\cdot}023411 e^2.$$

It is readily seen that the increase in the moment of inertia over that of the critical Jacobian, which has been called Δk^2, will be a linear function of ζ and e^2, say

$$\mu\zeta + n e_J^2,$$

Hence the excess of my angular momentum over that of DARWIN, will be

$$\mu\zeta_{J-D}.$$

The value of μ is easily obtained by allowing e^2 to vanish in the analysis given on p. 72 of my previous paper. The quantity μ then appears as the rate of increase of k^2 as we pass along the Jacobian series of ellipsoids. The general value of k^2, in the notation there used,* is

$$k^2 = \tfrac{1}{5}(a'^2 + b'^2) = \tfrac{1}{5}a^2(1 - 12{\cdot}71347\zeta) + \tfrac{1}{5}b^2(1 + 9{\cdot}20894\zeta),$$

whence

$$\mu = \frac{\partial (k^2)}{\partial \zeta} = -7{\cdot}84851.$$

The excess of my value of k^2 over that of DARWIN ought accordingly to be

$$\mu\zeta_{J-D} = -0{\cdot}183060 e_J^2.$$

* I have changed the sign of ζ, which, by an oversight, had been taken with the opposite sign in my formulæ as printed in the previous paper.

My value of Δk^2 having been seen to be $-0{\cdot}079156e_J^2$, it follows that DARWIN'S ought to be

$$\Delta k^2 = 0{\cdot}10390e_J^2 = 0{\cdot}13317e_D^2$$

so that his value of k^2 as deduced from my calculations ought to be

$$k^2 = 0{\cdot}8441\ (1+0{\cdot}15777e_D^2).$$

In point of fact the value actually given by DARWIN was

$$k^2 = 0{\cdot}8441\ (1+0{\cdot}157786e_D^2).$$

It appears, then, that DARWIN'S moment of momentum agrees exactly with mine, as it ought, except for the difference introduced by the different values we have taken for n''. But, besides showing this, the calculations just given provide a check on the accuracy of the computations of both of us. Although our figures, as far as the second order, have been calculated by very widely different methods, their moments of momentum have been found to agree very closely.*

21. The moment of momentum in the cylindrical problem was announced in my two-dimensional paper to increase with increasing e^2.

On repeating the computations of this paper, I find that the coefficients in the equation of the surface were correctly given, but the final computation of k^2 was erroneous.† The corrected formula becomes

$$k^2 = k_o^2\ (1-0{\cdot}1679e^2).$$

The Stability Criterion.

22. We have now found for the pear-shaped figure of equilibrium,

$$\frac{\omega^2}{2\pi\rho} = 0{\cdot}14200\ (1+0{\cdot}05227e^2)$$

$$k^2 = 0{\cdot}8441\ (1-0{\cdot}09378e^2)$$

* It may perhaps be added that before I had discovered this check on my computations, I had calculated the mass of my pear-shaped figure by direct integration, using the method of § 19. The total mass ought of course to come to exactly M. I find that the terms in α^2, $\alpha\beta$ ... result in an increase of mass $0{\cdot}0689519\mathrm{M}e^2$, while those in L, M, N ... s balance this with a decrease $-0{\cdot}0689514\mathrm{M}e^2$.

† The error is in the very last stage of all; the value of $\frac{1}{2}\gamma$ on p. 95 ought to read

$$\tfrac{1}{2}\gamma = 20{\cdot}25\alpha^6 - 118{\cdot}93\alpha^4 + 41{\cdot}071\alpha^2 + 229{\cdot}51,$$

and this leads to the formula

$$k^2 = k_0^2\,(1-167{\cdot}88\theta^2) = k_0^2\,(1-0{\cdot}16788e^2).$$

I have also calculated k^2 by the method of § 19 of the present paper, and found

$$k^2 = k_0^2\,(1-0{\cdot}16791e^2).$$

It follows that the moment of momentum $\mathbf{M}$ is given by

$$\mathbf{M} = \mathbf{M}_0 (1 - 0\cdot 06765 e^2).$$

Thus $\mathbf{M} < \mathbf{M}_0$, so that the moment of momentum decreases as we pass along the series of pear-shaped figures, and this series is therefore unstable.

Summary and Discussion of Results.

23. Throughout the present paper, and my previous paper on the same subject, the critical Jacobian ellipsoid which bifurcates into the pear-shaped series of figures of equilibrium, has been taken to be

$$f \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0.$$

Any adjacent figure, whether of equilibrium or not, may be supposed to be

$$f + \phi = 0$$

where ϕ is a function of x, y, z in which the coefficients are numerically small. For special values of ϕ, this figure will be one of equilibrium. So long as we consider only figures which differ infinitesimally from $f = 0$, all the possible figures of equilibrium form a linear series, and ϕ is of the form

$$\phi = e\mathrm{P}, \quad \ldots \ldots \ldots \quad (108)$$

where P is a function of x, y, z and e is a parameter which must be so small that e^2 can be neglected.

In the previous paper it was shown that as soon as e^2 is taken into account, there must be supposed to be a doubly infinite series of figures of equilibrium. The general form of ϕ is

$$\phi = e\mathrm{P} + e^2\mathrm{Q} + \zeta\mathrm{Q}^1 \quad \ldots \ldots \ldots \quad (109)$$

where ζ is a second parameter of the same order of magnitude as e^2, but capable of varying quite independently of e^2. The value of $\dfrac{\omega^2}{2\pi\rho}$ for this figure of equilibrium is greater by ζ than the value for the critical Jacobian. The possible figures of equilibrium may be thought of as lying inside a rectangle having e, ζ as rectangular co-ordinates.

In the present paper I have carried the investigation as far as e^3, and find that the value of ϕ as far as third-order terms must be of the form

$$\phi = e\mathrm{P} + e^2\mathrm{Q} + \zeta\mathrm{Q}^1 + e^3(\mathrm{R} + \mathrm{KP}) \quad \ldots \ldots \ldots \quad (110)$$

where R is a new function of x, y, z and K is a constant. At first sight K appears to be at our disposal, for if we replace the parameter e by a new parameter $e + \theta e^3$, we can vary K as much as we please. But on examining the problem in detail it is found that K is always infinite except for one special value of ζ. For this special value of ζ we can eliminate K altogether by a new choice of parameter. But for other values of ζ, our solution is only valid if $e^3\text{K}$ is small, and, replacing $e^3\text{K}$ by a new parameter e^1, the solution reduces to

$$\phi = 0 \,.\, \text{P} + 0 \,.\, \text{Q} + e^1\text{P}$$

and so returns to the original first-order solution (108).

Thus, except for one special value of ζ, it is impossible to carry the linear series outside the second order rectangle; if we attempt to do so, the solution keeps lapsing back into the first-order solution.

In the previous paper I gave reasons for believing that Sir G. DARWIN, in his well-known investigation of this problem, had introduced a spurious equation of equilibrium. This extra equation could only be satisfied by assigning to ζ a special value, namely

$$\zeta = -0{\cdot}015988e^2.$$

Sir G. DARWIN accordingly gave this value to ζ, so that the value of ω^2 decreased on passing along his series of pear-shaped figures, and, assuming this value for ζ, he showed his series to be stable.

But the investigation of the previous paper showed that there was no need to assign this special value to ζ, and the present investigation has further shown that with this value of ζ it is impossible to extend the series beyond second-order terms at all. There is only one value of ζ which leads to a real linear series of configurations, and this is shown in the present paper to be

$$\zeta = +0{\cdot}007423e^2.$$

Thus as we pass along the true linear series ω^2 continually increases. The angular momentum is however found to decrease, so that the pear-shaped figure is shown to be unstable.

24. The amount of computation involved in the problem has proved to be very great, and as the whole question of stability or instability depends on the sign of a single term at the end of all this computation, the question of numerical accuracy becomes one of great importance.

The difference between my second-order figure and that of DARWIN arises solely from the difference in the value of ζ. The moment of inertia of such figures is a linear function of ζ, and a very simple calculation gives the rate at which it ought to vary with ζ. Allowing for this difference in ζ, I find that my computations give for the moment of inertia of DARWIN'S figure (in terms of DARWIN'S parameter e_{D}),

$$\text{M}k_0^2(1+0{\cdot}15777e_{\text{D}}^2),$$

while DARWIN calculated as the value of the same quantity

$$\text{M}k_0^2(1+0{\cdot}157786e_{\text{D}}^2).$$

Since two independent sets of computations, conducted by entirely different methods, have been found to lead to precisely the same result, it seems highly probable that this result is accurate. The agreement just mentioned may reasonably be regarded as guaranteeing the accuracy of all the second-order computations, both of DARWIN and myself.

The actual criterion of stability, however, depends on the value assigned to ζ, and since this depends in turn on the third-order terms, no check by comparison with DARWIN'S work is possible. Some support is given to my value of ζ by comparison with a parallel investigation of the "Equilibrium of Rotating Cylinders," which I published some years ago. Adjusting the parameters so that e shall have, as closely as possible, the same physical interpretation in the two problems, I find for the factor expressing the increase of ω^2 as we pass along the series of pear-shaped figures:

$$1+0{\cdot}05227e^2 \text{ for the three-dimensional problem,}$$

$$1+0{\cdot}0513e^2 \text{ for the two-dimensional problem.}$$

Apart from this, the checks I have used in the present paper are such that I believe it would have been very difficult for any error to escape detection.

25. The main object of the paper is achieved as soon as the pear-shaped figure is shown to be unstable. It is at the same time of interest to examine the bearings of this result on the wider question of which it is a part.

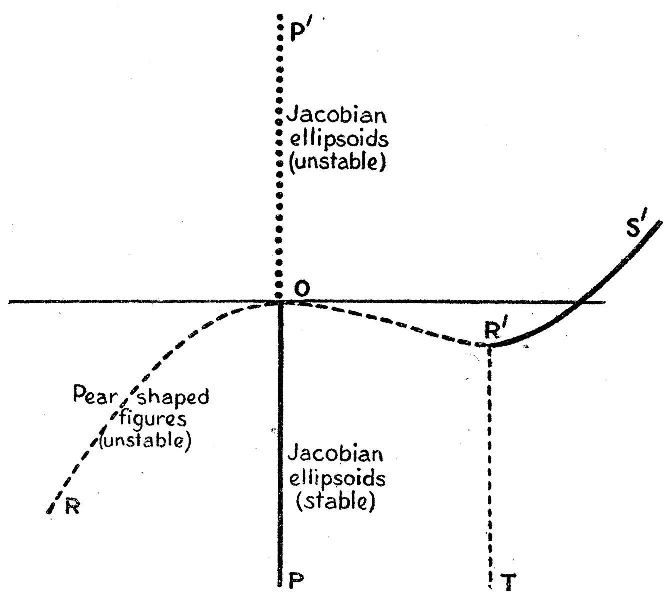

Fig. 1.

In fig. 1 let PP′ represent the series of Jacobian ellipsoids, the part PO (drawn thick) representing the stable part of the series, and the part OP′ (drawn dotted) representing the unstable part, so that O is the point of bifurcation. Let a diagram be drawn about this line having the angular momentum always represented by the vertical co-ordinates, so that for instance all systems, whether in equilibrium or not, which have the same angular momentum as the critical Jacobian ellipsoid, must be represented in the horizontal plane through O (this plane having as many "dimensions" as are necessary). In this diagram the pear-shaped series of figures will lie below this plane in the neighbourhood of O.

There are two *a priori* alternatives, represented on the right-hand and left-hand sides of fig. 1. In the first place, it is possible that after passing a certain distance OR′ along the pear-shaped series with decreasing angular momentum, we come to a region R′S′ in which the angular momentum again increases. Any horizontal line in the diagram ought, on the principle of stable and unstable configurations of equilibrium occurring alternately, to meet stable and unstable branches of the linear series alternately. Thus the branch R′S′ ought to be stable, so that R′ would be a point of bifurcation on this series, and the other series through R′, say R′T, would be unstable.

The alternative possibility is that the pear-shaped series of figures proceeds continually downwards in the diagram, so that its angular momentum continually decreases.

26. Either of these two possibilities removes a difficulty to which Sir G. DARWIN has drawn attention.* In what is commonly referred to as ROCHE'S problem an attempt is made to discover the other end of the pear-shaped series of figures, on the supposition that this other end represents two detached masses revolving about one another. Two such series of figures appear to exist†; in one the satellite is spheroidal except for tidal distortion, while in the other it is ellipsoidal. The former series has been shown to be stable, the latter unstable.

As the angular momentum decreases on passing along these series, the distance between the two masses also decreases until a point is reached at which the two series coalesce, the configuration of bifurcation being one in which the radius vector from the centre of the primary to that of the satellite is equal to 2·457 radii of the primary.‡ If the distance between the masses is decreased still further, the remaining configurations form an unstable series. Sir G. DARWIN found a difficulty in the instability of this series, since he believed it to be the far end of the pear-shaped series which he thought stable. We now see, however, that this series may, without change of stability, join up with either the series TR′ on the right-hand of our diagram, or the series RO on the left.

* 'Coll. Works,' III., pp. 515–524.

† ROCHE'S problem has only been solved strictly by imposing sphericity on the primary and assuming the satellite to be infinitesimal. Sir G. DARWIN'S work ('Coll. Works,' III., p. 436), leaves little room for doubt that ROCHE'S result may be extended in the way I have stated.

‡ If the satellite is not infinitesimal, the radius vector depends on the ratio of the masses, but always lies between the narrow limits 2·457 and 2·514 times the radius of the primary (see DARWIN, *loc. cit.*, p. 507).

27. We must now consider what motion is to be expected in a Jacobian ellipsoid which has reached the point of bifurcation at which instability sets in. POINCARÉ remarks* that if the pear-shaped figure proved to be unstable, "la masse fluide devrait le dissoudre par un cataclysme subit."

After reaching the point O in our diagram, the mass cannot move along the pear-shaped series, since this would involve a decrease of angular momentum. It may be thought of as moving along the unstable branch OP′ of the series of Jacobian ellipsoids for an infinitesimal time until some slight disturbance brings its instability into play.

Now of all the vibrations of this figure, it is known that one only is unstable, namely that corresponding to the third zonal harmonic of the ellipsoid. The initial motion of the figure must then be one in which the displacement at every point of the surface is proportional to the third zonal harmonic.

Thus the fluid begins by describing exactly the pear-shaped series, but as soon as the changes in angular momentum become appreciable, it leaves this series, and passes through a series of configurations represented in the region above O in fig. 1. These may at first be thought of as lying parallel to the pear-shaped series, but above O.

If there is a stable branch such as R′S′ which ultimately passes above O, it is conceivable that the series of non-equilibrium configurations might ultimately coalesce with the series of equilibrium configurations R′S′, and the motion would be continued along this series. In this case, M. POINCARÉ's "cataclysme subit" would consist in a jump from the stable series PO to the stable series R′S′.

Judging from the results of my parallel investigation on the configurations of rotating cylinders, this possibility does not seem at all likely. It is, I think, much more probable that the pear-shaped series lies like the series OR in my figure.

In this case also the liquid would move through a series of configurations which would initially be close to the series OR, but would get continually futher removed from configurations of equilibrium. The protuberance resulting from the initial third harmonic displacement would develop in a manner somewhat similar to that of the pear-shaped figure, but as the motion would necessarily be possessed of a considerable amount of kinetic energy, the phenomenon would be a dynamical and not a statical one. If the configuration represented at O in fig. 1 is the highest stable configuration possible for a single mass of liquid, this kinetic process can end in only one way, namely in the separation of the mass of liquid into two parts. As the third harmonic displacement develops, the region of the pear which moves with greatest velocity is known to be the extreme end of the protuberance. It is, therefore, natural to suppose that this part of the figure would be shot away first. Moreover as the departure from a figure of equilibrium is probably pretty pronounced before the separation takes place, it is likely that the mass in question will be shot away with a considerable velocity.

* Letter to Sir G. DARWIN in the latter's 'Coll. Works,' III., p. 315.

After this projectile has left the main mass, the angular momentum of the latter (measured of course per unit mass) will be reduced, and as the tidal influence of the newly-born satellite is gradually withdrawn, the primary may settle down to a state of stable equilibrium in which its figure is again that of a Jacobian ellipsoid. A cycle of processes such as this might very conceivably constitute the life-history of a rotating body after once it had passed the critical state represented by the point of bifurcation on the Jacobian series. There seems to be no reason why the protuberance should always develop at, and be shot off from, the same end of the Jacobian ellipsoid.

Thus it appears that the instability of the pear-shaped figure leads us to contemplate a series of processes of much the same nature as would have been expected if the pear had proved to be stable, except that we are now led to assign a much shorter time to these processes. The problem has ceased to be one of statics and has become one of dynamics; the phenomenon is no longer one of slow secular change but of collapse and explosion. The mechanism of planetary birth which is now suggested is so rapid that there need be no difficulty in supposing a planet to have been through the cycle several times; had the pear proved to be stable the cycle would probably have been so slow as to create a real difficulty.

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